

A PROJECTIVE-TO-CONFORMAL FEFFERMAN-TYPE CONSTRUCTION

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ABSTRACT. We study a Fefferman-type construction based on the inclusion $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. The construction associates a split-signature (n, n) -conformal spin structure to a projective structure of dimension n . We are able to show in the general (curved) situation the existence of a canonical (pure) twistor spinor and a light-like conformal Killing field on the constructed conformal space. We obtain a complete characterisation of the so-constructed conformal spaces in terms of this induced geometric data and an integrability-condition on the Weyl curvature. The Fefferman-type construction presented here is an alternative approach to study a conformal version of Patterson-Walker metrics as discussed in recent work by Dunajski-Tod and we relate the different viewpoints in an appendix.

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1. INTRODUCTION

In *conformal geometry* the geometric structure is given by an equivalence class of pseudo-Riemannian metrics: two metrics g and \hat{g} are considered to be equivalent if they differ by a positive smooth rescaling, $\hat{g} = e^{2f} g$. In *projective geometry* the geometric structure is given by an equivalence class of torsion-free affine connections: two connections D and \hat{D} are considered as equivalent if they share the same geodesics (as unparametrised curves). The present paper investigates a geometric construction that relates these

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two differential geometric structures. It produces an interesting conformal class of split-signature metrics coming from a projective class of connections. The main objective of the present work is to establish a local characterisation of the resulting conformal structures: We will identify properties of split-signature conformal structures that guarantee local equivalence to a conformal structure which is obtained from a projective structure via the geometric construction.

Note that while conformal and projective structures both determine a corresponding class of affine connections, neither of them induces a single distinguished connection on the tangent bundle. Instead, both structures have canonically associated *Cartan connections* that govern the respective geometries and encode prolonged geometric data of the respective structures. It is thus natural to study a procedure that associates a conformal structure to a projective structure in the framework of Cartan geometries, namely, by identifying it as an instance of a *Fefferman-type construction* as formalised in [Čap06, ČS09]. This viewpoint has proven to be effective in recent years for determining the special properties of a number of geometric spaces constructed in this manner: The original Fefferman construction [Fef76], which produces a canonical conformal structure on a circle bundle over a CR-manifold, and the local characterisation of these structures due to Sparling (cf. [Gra87]) was treated in this framework in [ČG08, ČG10]. Nurowski's [Nur05] and Bryant's [Bry06] conformal structures that are associated with certain types of generic distributions were studied by this approach in [HS09] and [HS11]. A Fefferman-type construction of conformal structures from quaternionic contact structures was investigated in [Alt10].

The original motivation for studying the particular Fefferman-type construction of this paper comes from two sources. The first one is work by Dunajski and Tod, [DT10], and Dunajski and West, [DW08]. These authors adapt the Riemannian extensions of affine structures as introduced by Patterson and Walker [PW52, Wal54] to a projectively invariant setting. The second source that motivated our investigations is a paper by Nurowski and Sparling, [NS03], which presents a construction of conformal structures of signature $(2, 2)$ using Cartan connections. In the language of this paper it can be understood as a Fefferman-type construction from 3-dimensional Lagrangean contact structures to conformal structures, and a special case yields the projective to conformal construction in the lowest admissible dimension. A generalisation of this approach to higher dimensions was recently presented in [Nur12]. We will clarify the relation between the different approaches, i.e. Riemannian extensions and Cartan connections, respectively, in the appendix of the present paper. Furthermore, a separate article, [HSSTZ], will be devoted to the local characterisation of the induced conformal structures from the viewpoint of a projectively invariant Walker-type construction.

Let us now remark on the technical aspects of the present paper and discuss the main results. In this article we investigate a Fefferman-type construction based on an inclusion $SL(n+1) \hookrightarrow Spin(n+1, n+1)$, associating a split-signature (n, n) conformal spin structure to an oriented n -dimensional projective structure. If $n = 2$ the induced conformal Cartan connection is

normal and the usual consequences of a conformal holonomy reduction can be derived. However, for $n \geq 3$, the induced conformal Cartan connection is shown to be normal if and only if the original projective structure was already flat, Proposition 4.3. This fact immediately poses problems for the goal of relating the original projective and the induced conformal structure. To obtain information on the conformal structure it is thus necessary to understand how the normal conformal Cartan connection differs from the induced one. By analysing the modification, we derive strong restrictions on the form of the normalised conformal Cartan connection. These imply, in particular, that the induced conformal spin structure is endowed with a solution of the twistor spinor equation. Further, it is shown that it admits a light-like conformal Killing field and that an integrability condition on the Weyl tensor holds, see Proposition 5.9. Finally, Theorem 5.16 and Theorem 6.8, establish a local characterisation of conformal structures induced from projective structures via the Fefferman-type construction in terms of these conformal data.

Outlook. A number of geometric overdetermined differential equations in projective and conformal geometry are defined by *first BGG-operators*, cf. [ČSS01, CD01]. For example, in projective geometry the question whether there exists a Ricci-flat connection in a given projective class, and in conformal geometry, whether there exists an Einstein metric in a given conformal class, are governed by BGG-operators. It is now interesting to ask whether solutions of certain projective equations determine solutions of certain conformal equations, and vice versa. The results of the present article form the basis of our investigations of such *BGG-relations*, which are the subject of a follow-up article.

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2. BACKGROUND ON PARABOLIC GEOMETRIES

This section surveys basic ideas and concepts of parabolic geometries both in general and concrete, namely, for projective and conformal structures. For later use, explicit formulae for several projective and conformal tractor bundles are collected in respective subsections. The standard reference for most of the background material presented here is [ČS09].

2.1. Cartan- and parabolic geometries. Let G be a Lie group with Lie algebra \mathfrak{g} and $P \subset G$ a closed subgroup with Lie algebra \mathfrak{p} . A *Cartan geometry* (\mathcal{G}, ω) of type (G, P) over a smooth manifold M consists of a P -principal

bundle $\mathcal{G} \rightarrow M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, i.e., a \mathfrak{g} -valued 1-form on \mathcal{G} that (i) is P -equivariant, (ii) maps each fundamental vector field ζ_X to its generator $X \in \mathfrak{p}$, and (iii) defines a linear isomorphism $\omega(u) : T_u\mathcal{G} \rightarrow \mathfrak{g}$ for each $u \in \mathcal{G}$. The canonical principal bundle $G \rightarrow G/P$ endowed with the Maurer–Cartan form constitutes the *homogeneous model* for Cartan geometries of type (G, P) .

The *curvature* of a Cartan connection ω is the 2-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined as

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. Since the curvature is strictly horizontal, it is equivalently encoded in the *curvature function*, the P -equivariant mapping $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ given by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)). \quad (1)$$

The curvature is a complete obstruction to a local equivalence with the homogeneous model. If the image of κ is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$, then the Cartan geometry is called *torsion-free*.

A *parabolic geometry* is a Cartan geometry of type (G, P) , where G is a semisimple Lie group and $P \subset G$ is a parabolic subgroup; a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is parabolic if and only if its maximal nilpotent ideal, called nilradical \mathfrak{p}_+ , coincides with the orthogonal complement \mathfrak{p}^\perp of $\mathfrak{p} \subset \mathfrak{g}$ with respect to the Killing form. In particular, this yields an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P -modules. The quotient $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$ is called the Levi factor; it is reductive and decomposes as $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0)$ into a semisimple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ and the center $\mathfrak{z}(\mathfrak{g}_0)$. An identification of \mathfrak{g}_0 with a subalgebra in \mathfrak{p} yields a grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$. The respective Lie groups are $G_0^{ss} \subset G_0 \subset P$ and $P_+ \subset P$ so that $P = G_0 \ltimes P_+$ and $P_+ = \exp(\mathfrak{p}_+)$.

In the case that the nilradical \mathfrak{p}_+ is abelian, the representation of \mathfrak{g}_0 on \mathfrak{p}_+ is irreducible. However if \mathfrak{p}_+ is not abelian, the previous grading of \mathfrak{g} may be refined; if k is the depth of the finest possible grading, then the parabolic geometry is called *$|k|$ -graded*. The grading of \mathfrak{g} induces a grading on $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \cong \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. A parabolic geometry is called *regular*, if the curvature function κ takes values only in the components of positive homogeneity. In particular, any torsion-free or $|1|$ -graded parabolic geometry is regular.

Given a \mathfrak{g} -module V , there is a natural \mathfrak{p} -equivariant map, the *Kostant co-differential* [Kos61],

$$\partial^* : \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V \rightarrow \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes V, \quad (2)$$

defining the Lie algebra cohomology of \mathfrak{p}_+ with values in V ; see e.g. [ČS09, Section 3.3.1] for the explicit form. For $V = \mathfrak{g}$, this gives rise to a natural normalisation condition: parabolic geometries satisfying $\partial^*(\kappa) = 0$ are called *normal*. The *harmonic curvature* κ_H of a normal parabolic geometry is the image of κ under the projection $\ker \partial^* \rightarrow \ker \partial^* / \text{im } \partial^*$. For regular and normal parabolic geometries, the entire curvature κ is completely determined just by κ_H .

2.1.1. Tractor bundles and connections. Every Cartan connection ω on $\mathcal{G} \rightarrow M$ naturally extends to a principal connection $\hat{\omega}$ on the G -principal bundle

$\hat{\mathcal{G}} := \mathcal{G} \times_P G \rightarrow M$, which further induces a linear connection $\nabla^\mathcal{V}$ on any associated vector bundle $\mathcal{V} := \mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ for a G -representation V . Bundles \mathcal{V} and connections $\nabla^\mathcal{V}$ arising in this way are called *tractor bundles* and *tractor connections*. The tractor connections induced by normal Cartan connections are called normal tractor connections.

In particular, for $V = \mathfrak{g}$ and the adjoint representation we obtain the *adjoint tractor bundle* $\mathcal{AM} := \mathcal{G} \times_P \mathfrak{g}$. The canonical projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p}$ and the identification $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ yield a bundle projection $\Pi : \mathcal{AM} \rightarrow TM$; the inclusion $\mathfrak{p}_+ \subset \mathfrak{g}$ and the identification $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ yield a bundle inclusion $T^*M \hookrightarrow \mathcal{AM}$. This allows us to interpret the Cartan curvature κ from (1) as a 2-form Ω on M with values in \mathcal{AM} . For a general G -representation V , its derivative $\mathfrak{g} \times V \rightarrow V$ induces a natural bundle map $\bullet : \mathcal{AM} \times \mathcal{V} \rightarrow \mathcal{V}$. The relation between the curvature $\Omega \in \Omega^2(M, \mathcal{AM})$ of the Cartan connection ω and the curvature $\Omega^\mathcal{V} \in \Omega^2(M, \text{End}(\mathcal{V}))$ of the induced tractor connection $\nabla^\mathcal{V}$ is given by

$$\Omega^\mathcal{V}(\xi, \eta)(s) = \Omega(\xi, \eta) \bullet s,$$

for any $\xi, \eta \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V})$.

The holonomy group of the principal connection $\hat{\omega}$ is by definition the *holonomy of Cartan connection* ω , i.e.

$$\text{Hol}(\omega) := \text{Hol}(\hat{\omega}) \subseteq G. \quad (3)$$

The holonomy group of a tractor connection $\nabla^\mathcal{V}$ coincides with the image of $\text{Hol}(\hat{\omega})$ under the defining representation $G \rightarrow GL(V)$. Parallel sections of tractor bundles always lead to reductions of holonomies.

2.1.2. BGG-operators and first BGG-equations. In [ČSS01], and later in a simplified manner in [CD01], it was shown that for a tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$ one can associate a sequence of differential operators, which are intrinsic to the given parabolic geometry (\mathcal{G}, ω) ,

$$\Gamma(\mathcal{H}_0) \xrightarrow{\Theta_0^\mathcal{V}} \Gamma(\mathcal{H}_1) \xrightarrow{\Theta_1^\mathcal{V}} \dots \xrightarrow{\Theta_{n-1}^\mathcal{V}} \Gamma(\mathcal{H}_n).$$

The operators $\Theta_k^\mathcal{V}$ are the *BGG-operators* and they operate between the sub-quotients $\mathcal{H}_k = \ker \partial^* / \text{im } \partial^*$ of the bundles of \mathcal{V} -valued k -forms, where $\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \rightarrow \Lambda^{k-1} T^*M \otimes \mathcal{V}$ denotes the bundle map induced by the Kostant co-differential (2). A key ingredient in their construction is the covariant exterior derivative $d^{\nabla^\mathcal{V}} : \Omega^k(M, \mathcal{V}) \rightarrow \Omega^{k+1}(M, \mathcal{V})$ given by the tractor connection $\nabla^\mathcal{V}$. We remark that $\Theta_k^\mathcal{V}$ form a complex if and only if the parabolic geometry (\mathcal{G}, ω) is locally flat.

The first BGG-operator $\Theta_0^\mathcal{V} : \Gamma(\mathcal{H}_0) \rightarrow \Gamma(\mathcal{H}_1)$ is constructed as follows. The bundle \mathcal{H}_0 is simply the quotient \mathcal{V}/\mathcal{V}' , where $\mathcal{V}' \subset \mathcal{V}$ is the sub-bundle corresponding to the largest P -invariant filtration component in the G -representation V . It turns out, there is a distinguished differential operator that splits the projection $\Pi_0 : \mathcal{V} \rightarrow \mathcal{H}_0$, namely, the *splitting operator*, which is the unique map $L_0^\mathcal{V} : \Gamma(\mathcal{H}_0) \rightarrow \Gamma(\mathcal{V})$ satisfying

$$\Pi_0(L_0^\mathcal{V}(\sigma)) = \sigma \quad \text{and} \quad \partial^*(d^{\nabla^\mathcal{V}} L_0^\mathcal{V}(\sigma)) = 0, \quad \text{for any } \sigma \in \Gamma(\mathcal{H}_0). \quad (4)$$

The latter condition allows to define the first BGG-operator by

$$\Theta_0^\mathcal{V} := \Pi_1 \circ d^{\nabla^\mathcal{V}} \circ L_0^\mathcal{V},$$

where $\Pi_1 : \ker \partial^* \rightarrow \Gamma(\mathcal{H}_1)$. The first BGG-operator defines an overdetermined system of differential equations on $\sigma \in \Gamma(\mathcal{H}_0)$, $\Theta_0^\mathcal{V}(\sigma) = 0$, which is termed the *first BGG-equation*. For the projective and conformal structures we discuss below, a number of interesting geometric equations is encoded as first BGG-equations.

2.1.3. Weyl structures and connections. A *Weyl structure* of a parabolic geometry (\mathcal{G}, ω) over M is a reduction of the P -principal bundle $\mathcal{G} \rightarrow M$ to the Levi subgroup $G_0 \subset P$; the corresponding G_0 -bundle is denoted by $\mathcal{G}_0 \rightarrow M$ and the embedding by $j : \mathcal{G}_0 \hookrightarrow \mathcal{G}$. The class of all Weyl structures, which is parameterised by one-forms on M , includes a particularly important subclass of *exact Weyl structures*, which is parameterised by functions on M : For $|1|$ -graded parabolic geometries, these correspond to further reductions of $\mathcal{G}_0 \rightarrow M$ just to the semisimple part G_0^{ss} of G_0 or, equivalently, to sections of the principal \mathbb{R}_+ -bundle $\mathcal{G}_0/G_0^{ss} \rightarrow M$. The latter bundle is called the *bundle of scales* and its sections are the *scales*.

For a Weyl structure $j : \mathcal{G}_0 \hookrightarrow \mathcal{G}$, the pullback of the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ may be decomposed according to the grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$:

$$j^*\omega = j^*\omega_- + j^*\omega_0 + j^*\omega_+.$$

The \mathfrak{g}_0 -part of $j^*\omega$ is a principal connection on the G_0 -bundle $\mathcal{G}_0 \rightarrow M$; it induces connections on all associated bundles, which are called (*exact*) *Weyl connections*. In particular, any exact Weyl connection preserves the corresponding scale. The induced Weyl connection on TM is denoted by D , the curvature of D is denoted by R . The \mathfrak{p}_+ -part of $j^*\omega$ is the so called *Schouten tensor* denoted by P ; it is interpreted as a one-form on M with values in T^*M , i.e. as a section of $T^*M \otimes T^*M$.

Likewise, we may decompose the pullback of the curvature. For normal $|1|$ -graded parabolic geometries we employ the following notation: The \mathfrak{g}_- -part of j^*K is the torsion, T , of the Weyl connection D , the \mathfrak{g}_0 - and \mathfrak{p}_+ -part of j^*K is the so called *Weyl curvature*, W , and *Cotton tensor*, Y , respectively. In the previous terms, they may be expressed as

$$W = R + \partial P, \tag{5}$$

$$Y = d^D P, \tag{6}$$

where ∂ is the bundle map induced by the standard differential that is adjoint to (2) and d^D is the covariant exterior derivative determined by D . Note that first non-zero tensor in the sequence (T, W, Y) is independent of the choice of Weyl structure as it corresponds to the lowest non-zero homogeneous component of the harmonic curvature κ_H . In the case of projective or conformal structures, the torsion T of any Weyl connection vanishes identically.

The G_0 -principal connection $j^*\omega_0$ induces a connection on any bundle associated to \mathcal{G}_0 , in particular, on any tractor bundle $\mathcal{V} = \mathcal{G} \times_P V = \mathcal{G}_0 \times_{G_0} V$. Denoting the induced connection by $D^\mathcal{V}$, the tractor connection may be

expressed as

$$\nabla^\mathcal{V} = \partial + D^\mathcal{V} + \mathbf{P}\bullet, \quad (7)$$

i.e. $\nabla_\xi^\mathcal{V} s = \xi \bullet s + D_\xi^\mathcal{V} s + \mathbf{P}(\xi) \bullet s$, for any $\xi \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V})$.

2.1.4. Notations and conventions for projective and conformal structures.

In order to distinguish various objects related to projective and conformal structures, the symbols referring to conformal data will always be endowed with tildes. To write down explicit formulae, we employ abstract index notation, cf. e.g. [PR84]. Furthermore, we will use different types of indices for projective and conformal manifolds. E.g., on a projective manifold M we write $\mathbb{E}_A := T^*M$, $\mathbb{E}^A := TM$, and multiple indices denote tensor products, as in $\mathbb{E}_A{}^B := T^*M \otimes TM$. Indices between squared brackets are skew, as in $\mathbb{E}_{[AB]} := \Lambda^2 T^*M$, and indices between round brackets are symmetric, as in $\mathbb{E}^{(AB)} := S^2 TM$. Analogously, on a conformal manifold \widetilde{M} we write $\widetilde{\mathbb{E}}_a := T^*\widetilde{M}$, $\widetilde{\mathbb{E}}^a := T\widetilde{M}$ etc.

By $\mathbb{E}(w)$ and $\widetilde{\mathbb{E}}[w]$ we denote the density bundle over M and \widetilde{M} , respectively, with specific standard and structure-adapted normalisation which will be introduced later. Tensor products with another natural bundles are denoted as $\mathbb{E}_A(w) := \mathbb{E}_A \otimes \mathbb{E}(w)$, $\widetilde{\mathbb{E}}_{[ab]}[w] := \widetilde{\mathbb{E}}_{[ab]} \otimes \widetilde{\mathbb{E}}[w]$, and the like.

Previously introduced superscripts for tractor-related objects, like $\nabla^{\widetilde{\mathcal{V}}}$, may be omitted or substituted by other markers that are more important in given context. Typically, we will use $\widetilde{\nabla}^{ind}$ and $\widetilde{\nabla}^{nor}$ to distinguish between tractor connections determined by the induced and the normal conformal Cartan connection, respectively.

2.2. Projective structures. Let M be a smooth manifold of dimension $n \geq 2$. A *projective structure* on M is given by a class, \mathbf{p} , of torsion-free projectively equivalent affine connections: two connections D and \hat{D} are projectively equivalent if they have the same geodesics as unparametrised curves. This is the case if and only if there is a one-form $\Upsilon_A \in \Gamma(\mathbb{E}_A)$ such that, for all $\xi^A \in \Gamma(\mathbb{E}^A)$,

$$\hat{D}_A \xi^B = D_A \xi^B + \Upsilon_A \xi^B + \Upsilon_P \xi^P \delta_A{}^B, \quad (8)$$

where $\delta_A{}^B$ is the Kronecker symbol for the identity map on the tangent bundle.

An *oriented projective structure* (M, \mathbf{p}) , which is a projective structure \mathbf{p} on an oriented manifold M , is equivalently encoded as a normal parabolic geometry of type (G, P) , where $G = \mathrm{SL}(n+1)$ and $P = \mathrm{GL}(n) \ltimes \mathbb{R}^{n*}$ is the stabiliser of a ray in the standard representation \mathbb{R}^{n+1} . The homogeneous model G/P is the standard projective sphere S^n .

Affine connections from the projective class \mathbf{p} are precisely the Weyl connections of the corresponding parabolic geometry. Exact Weyl connections are those $D \in \mathbf{p}$ which preserve a volume form — these are also known as *special* affine connections. In particular, a choice of $D \in \mathbf{p}$ reduces the structure group to $G_0 = \mathrm{GL}(n)$, if D is special, the structure group is further reduced to $G_0^{ss} = \mathrm{SL}(n)$.

For later purposes we now give explicit expressions of the main curvature quantities, cf. e.g. [Eas08, BEG94]. For $D \in \mathbf{p}$, the Schouten tensor is

determined by the Ricci curvature of D ; if D is special, then the Schouten tensor is

$$P_{AB} = \frac{1}{n-1} R_{PA}{}^P{}_B, \quad (9)$$

in particular, it is symmetric. The projective Weyl curvature and the Cotton tensor are

$$W_{AB}{}^C{}_D = R_{AB}{}^C{}_D + P_{AD}\delta^C_B - P_{BD}\delta^C_A, \quad (10)$$

$$Y_{CAB} = 2D_{[A}P_{B]C}. \quad (11)$$

Note that for $n = 2$, the Weyl curvature vanishes identically, hence the only obstruction to the local flatness is the Cotton tensor (in particular, it does not depend on the choice of Weyl structure).

Henceforth, we use a suitable normalisation of densities so that the line bundle associated to the canonical one-dimensional representation of P has projective weight -1 . Hence, comparing with the usual notation, the *density bundle of projective weight w* , denoted by $\mathbb{E}(w)$, is just the bundle of ordinary $\left(\frac{-w}{n+1}\right)$ -densities. As an associated bundle to $\mathcal{G} \rightarrow M$, $\mathbb{E}(w)$ corresponds to the 1-dimensional representation of P given by

$$\mathrm{GL}(n) \ltimes \mathbb{R}^{n*} \rightarrow \mathbb{R}_+, \quad (A, X) \mapsto |\det(A)|^w. \quad (12)$$

2.2.1. The projective dual standard tractor bundle. The standard tractor bundle is the tractor bundle associated to the standard representation of $G = \mathrm{SL}(n+1)$. The dual bundle is denoted by \mathcal{T}^* , i.e. $\mathcal{T}^* := \mathcal{G} \times_P \mathbb{R}^{n+1*}$. With respect to a choice of $D \in \mathfrak{p}$, it is identified with $\mathbb{E}_A(1) \oplus \mathbb{E}(1)$, which is schematically written as $\mathcal{T}^* = \begin{pmatrix} \mathbb{E}_A(1) \\ \mathbb{E}(1) \end{pmatrix}$. Note that $\Pi_0 : \mathcal{T}^* \rightarrow \mathbb{E}(1) \cong \mathcal{H}_0^{\mathcal{T}^*}$ is the projectively invariant projection to the lowest slot. In these terms, the tractor connection is given by

$$\nabla_C^{\mathcal{T}^*} \begin{pmatrix} \varphi_A \\ \sigma \end{pmatrix} = \begin{pmatrix} D_C \varphi_A + P_{CA} \sigma \\ D_C \sigma - \varphi_C \end{pmatrix}.$$

It turns out, the first splitting operator is

$$L_0^{\mathcal{T}^*} : \Gamma(\mathbb{E}(1)) \rightarrow \Gamma(\mathcal{T}^*), \quad \sigma \mapsto \begin{pmatrix} D_A \sigma \\ \sigma \end{pmatrix}.$$

2.3. Conformal spin structures. Let \widetilde{M} be a smooth manifold of dimension $2n \geq 4$. A *conformal structure* of signature (n, n) on \widetilde{M} is given by a class, \mathbf{c} , of conformally equivalent pseudo-Riemannian metrics of signature (n, n) : two metrics g and \hat{g} are conformally equivalent if $\hat{g} = f^2 g$ for a nowhere-vanishing smooth function f on \widetilde{M} . It may be equivalently described as a reduction of the principal fibre bundle over \widetilde{M} to the structure group $\mathrm{CO}(n, n) = \mathbb{R}_+ \times \mathrm{SO}(n, n)$. An *oriented conformal structure* of signature (n, n) is a conformal structure of signature (n, n) together with fixed orientations both in time-like and space-like directions, equivalently, a reduction of the principal fibre bundle to the group $\mathrm{CO}_o(n, n) = \mathbb{R}_+ \times \mathrm{SO}_o(n, n)$, the connected component of the identity. A *conformal spin structure* $(\widetilde{M}, \mathbf{c})$ of signature (n, n) is a reduction of the principal fibre bundle over \widetilde{M} to

the structure group $\text{CSpin}(n, n) = \mathbb{R}_+ \times \text{Spin}(n, n)$, the 2-fold covering of $\text{CO}_o(n, n)$.

A conformal spin structure of signature (n, n) structure is equivalently encoded as a normal parabolic geometry of type (\tilde{G}, \tilde{P}) , where $\tilde{G} = \text{Spin}(n+1, n+1)$ and $\tilde{P} = \text{CSpin}(n, n) \ltimes \mathbb{R}^{n, n*}$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{n+1, n+1}$. The homogeneous model \tilde{G}/\tilde{P} is the product $S^n \times S^n$ of two standard spheres (with round metrics of opposite signs).

Any choice of $g \in \mathfrak{c}$ yields the canonical torsion-free Levi-Civita connection \tilde{D} , $\tilde{D}g = 0$, which is an exact Weyl connection of the corresponding parabolic geometry. A general Weyl connection is a torsion-free affine connection \tilde{D} such that $\tilde{D}g \in \mathfrak{c}$ for any $g \in \mathfrak{c}$. A choice of Weyl connection reduces the structure group to $\tilde{G}_0 = \text{CSpin}(n, n)$. If it is a Levi-Civita connection of a metric from the conformal class, the structure group is further reduced to $\tilde{G}_0^{ss} = \text{Spin}(n, n)$.

Now we briefly introduce the main curvature quantities of conformal structures, cf. e.g. [Eas96]. For $g \in \mathfrak{c}$, the *Schouten tensor*,

$$\tilde{P} = \tilde{P}(g) = \frac{1}{2n-2}(\text{Ric}(g) - \frac{\text{Sc}(g)}{2(2n-1)}g)$$

is a trace modification of the Ricci curvature $\text{Ric}(g)$ by a multiple of the scalar curvature $\text{Sc}(g)$. The full trace of the Schouten tensor is denoted $\tilde{J} = g^{pq}\tilde{P}_{pq}$. The conformal Weyl curvature and the Cotton tensors are

$$\begin{aligned}\tilde{W}_{ab}{}^c{}_d &= \tilde{R}_{ab}{}^c{}_d - 2\delta_{[a}^c\tilde{P}_{b]d} + 2g_{d[a}\tilde{P}_{b]}^c, \\ \tilde{Y}_{cab} &= 2\tilde{D}_{[a}\tilde{P}_{b]c}.\end{aligned}$$

For $\dim \tilde{M} = 2n \geq 4$, the Weyl curvature is the complete obstruction to local flatness of the conformal structure.

As for projective structures, we will employ a suitable parametrisation of densities so that the canonical 1-dimensional representation of \tilde{P} has conformal weight -1 . Hence, the *density bundle of conformal weight w* , denoted as $\tilde{\mathbb{E}}[w]$, is just the bundle of ordinary $(\frac{-w}{2n})$ -densities. As an associated bundle to the Cartan bundle $\tilde{\mathcal{G}} \rightarrow \tilde{M}$, it corresponds to the 1-dimensional representation of \tilde{P} given by

$$(\mathbb{R}_+ \times \text{Spin}(n, n)) \ltimes \mathbb{R}^{2n*} \rightarrow \mathbb{R}_+, \quad (a, A, Z) \mapsto a^{-w}. \quad (13)$$

In particular, the conformal structure may be seen as a section of $\tilde{\mathbb{E}}_{(ab)}[2]$, which is called the *conformal metric* and denoted by \mathbf{g}_{ab} . Contracting with \mathbf{g}_{ab} we have an identification $\tilde{\mathbb{E}}^a \cong \tilde{\mathbb{E}}_a[2]$.

The spin bundles corresponding to the irreducible spin representations of $\text{Spin}(n, n)$ are denoted by \tilde{S}_+ and \tilde{S}_- , and $\tilde{S} = \tilde{S}_+ \oplus \tilde{S}_-$. We employ the weighted conformal gamma matrix $\gamma \in \Gamma(T^*\tilde{M} \otimes (\text{End } \tilde{S})[1])$ so that $\gamma_p\gamma_q + \gamma_q\gamma_p = -2\mathbf{g}_{pq}$. For $\xi \in \mathfrak{X}(\tilde{M})$ and $\chi \in \Gamma(\tilde{S})$, the Clifford multiplication of ξ on χ is then written as $\xi \cdot \chi = \xi^p\gamma_p\chi$.

2.3.1. *The conformal standard tractor bundle.* This is the associated bundle $\tilde{\mathcal{T}} := \tilde{\mathcal{G}} \times_{\tilde{P}} \mathbb{R}^{n+1, n+1}$ with respect to the standard representation. With respect to $g \in \mathbf{c}$, it decomposes as

$$[\tilde{\mathcal{T}}]_g = \begin{pmatrix} \tilde{\mathbb{E}}[-1] \\ \tilde{\mathbb{E}}_a[1] \\ \tilde{\mathbb{E}}[1] \end{pmatrix} \ni \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = [s]_g \quad (14)$$

which also shows notation for slots of a section $s \in \Gamma(\tilde{\mathcal{T}})$. Note the projection to the lowest slot $\Pi_0^{\tilde{\mathcal{T}}} : \tilde{\mathcal{T}} \rightarrow \tilde{\mathbb{E}}[1] = \mathcal{H}_0^{\tilde{\mathcal{T}}}$ is the conformally invariant. Further, $\tilde{\mathcal{T}}$ carries invariant tractor metric $\mathbf{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which is compatible with the standard tractor connection

$$[\tilde{\nabla}_c^{\tilde{\mathcal{T}}} \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix}]_g = \begin{pmatrix} \tilde{D}_c \rho - \tilde{\mathbf{P}}_c^b \varphi_b \\ \tilde{D}_c \varphi_a + \sigma \tilde{\mathbf{P}}_{ca} + \rho \mathbf{g}_{ca} \\ \tilde{D}_c \sigma - \varphi_c \end{pmatrix} \quad (15)$$

where \tilde{D} denotes the Levi-Civita connection of the metric g . The curvature of $\tilde{\nabla}^{\tilde{\mathcal{T}}}$ in the corresponding block form is

$$\tilde{\Omega}_{ab} = \begin{pmatrix} 0 & -\tilde{Y}_{dab} & 0 \\ 0 & \tilde{W}_{ab}^c{}^d & \tilde{Y}_{ab}^c \\ 0 & 0 & 0 \end{pmatrix}.$$

The BGG-splitting operator of $\tilde{\mathcal{T}}$ is

$$L_0^{\tilde{\mathcal{T}}} : \Gamma(\tilde{\mathbb{E}}[1]) \rightarrow \Gamma(\tilde{\mathcal{T}}), \quad \sigma \mapsto \begin{pmatrix} \frac{1}{2n}(-\tilde{D}^p \tilde{D}_p - \tilde{J})\sigma \\ \tilde{D}_a \sigma \\ \sigma \end{pmatrix}. \quad (16)$$

2.3.2. *The spin tractor bundle.* This is the associated bundle $\tilde{\mathcal{S}} := \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta^{n+1, n+1}$, where $\Delta^{n+1, n+1}$ is the spin representation of $\tilde{G} = \text{Spin}(n+1, n+1)$. Since we work in even signature, this decomposes into irreducibles $\Delta^{n+1, n+1} = \Delta_+^{n+1, n+1} \oplus \Delta_-^{n+1, n+1}$; the corresponding bundles are denoted by $\tilde{\mathcal{S}}_{\pm} = \tilde{\mathcal{G}} \times_{\tilde{P}} \Delta_{\pm}^{n+1, n+1}$. Under a choice of $g \in \mathbf{c}$ the spin tractor bundles decompose as follows: $[\tilde{\mathcal{S}}_{\pm}]_g = \begin{pmatrix} \tilde{S}_{\pm}[-\frac{1}{2}] \\ \tilde{S}_{\pm}[\frac{1}{2}] \end{pmatrix}$, where \tilde{S}_{\pm} are the natural spin

bundles as before. Note that $\Pi_0^{\tilde{\mathcal{S}}} : \tilde{\mathcal{S}}_{\pm} \rightarrow \tilde{S}_{\pm}[\frac{1}{2}] \cong \mathcal{H}_0^{\tilde{\mathcal{S}}_{\pm}}$ is the conformally invariant projection to the lowest slot. Sections of $\tilde{\mathcal{S}}_{\pm}$ thus have the form $\begin{pmatrix} \tau \\ \chi \end{pmatrix} \in \tilde{\mathcal{S}}_{\pm}$.

The Clifford action of the conformal standard tractor bundle $\tilde{\mathcal{T}}$ on $\tilde{\mathcal{S}}$ is given by

$$\begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\varphi_a \gamma^a \tau + \sqrt{2} \rho \chi \\ \varphi_a \gamma^a \chi - \sqrt{2} \sigma \tau \end{pmatrix}, \quad (17)$$

cf. [Ham09, Ham12]. $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_+ \oplus \tilde{\mathcal{S}}_-$ carries the spin tractor connections that are induced from the standard tractor connection on $\tilde{\mathcal{T}}$:

$$[\tilde{\nabla}_c^{\tilde{\mathcal{S}}} \begin{pmatrix} \tau \\ \chi \end{pmatrix}]_g = \begin{pmatrix} \tilde{D}_c \tau + \frac{1}{\sqrt{2}} \tilde{\mathbf{P}}_{cp} \gamma^p \chi \\ \tilde{D}_c \chi + \frac{1}{\sqrt{2}} \gamma_c \tau \end{pmatrix}.$$

The BGG-splitting operator of $\tilde{\mathcal{S}}_{\pm}$ is

$$L_0^{\tilde{\mathcal{S}}_{\pm}} : \Gamma(S_{\pm}[\frac{1}{2}]) \rightarrow \Gamma(\tilde{\mathcal{S}}_{\pm}), \quad \chi \mapsto \begin{pmatrix} \frac{1}{\sqrt{2n}} \mathcal{D} \chi \\ \chi \end{pmatrix}. \quad (18)$$

Here

$$\mathcal{D} : \Gamma(S_{\pm}) \rightarrow \Gamma(S_{\mp}), \quad \mathcal{D} := \gamma^p \tilde{D}_p,$$

is the *Dirac* operator. It turns out $\mathcal{H}_1^{\tilde{\mathcal{S}}_{\pm}} \cong T^* \tilde{\mathcal{M}} \otimes S_{\pm}[\frac{1}{2}]$ and the first BGG-operator is

$$\Theta_0^{\tilde{\mathcal{S}}} : \Gamma(S_{\pm}[\frac{1}{2}]) \rightarrow \Gamma(T^* \tilde{\mathcal{M}} \otimes S_{\pm}[\frac{1}{2}]), \quad \Theta_0^{\tilde{\mathcal{S}}}(\chi) = \tilde{D}_a \chi + \frac{1}{2n} \gamma_a \mathcal{D} \chi.$$

This is the *twistor operator* (cf. e.g. [BFGK90]), which is alternatively described as the projection of the Levi-Civita derivative of a spinor to the kernel of Clifford multiplication. The kernel of the twistor operator is called the space of twistor spinors $\mathbf{Tw}(\mathbf{c})$, and $\Pi_0^{\tilde{\mathcal{S}}}$ induces an isomorphism of the space of $\tilde{\nabla}^{\tilde{\mathcal{S}}}$ -parallel sections of $\tilde{\mathcal{S}}$ with $\mathbf{Tw}(\mathbf{c})$ in $\Gamma(S[\frac{1}{2}])$.

2.3.3. The adjoint tractor bundle. This is the associated bundle $\mathcal{AM} := \tilde{\mathcal{G}} \times_{\tilde{\mathbf{P}}} \tilde{\mathbf{g}}$ with respect to the adjoint representation of \tilde{G} on $\tilde{\mathbf{g}} = \mathfrak{so}(n+1, n+1) \cong \Lambda^2 \mathbb{R}^{n+1, n+1}$. Henceforth we identify \mathcal{AM} with $\Lambda^2 \tilde{\mathcal{T}}$. The projection $\Pi_0^{\mathcal{AM}} : \mathcal{AM} \rightarrow \tilde{\mathbb{E}}_a[2] \cong \tilde{\mathbb{E}}^a \cong \mathcal{H}_0^{\mathcal{AM}}$ to the lowest slot is conformally invariant.

With respect to a $g \in \mathbf{c}$, an element $s \in \Gamma(\mathcal{AM}) = \Gamma(\Lambda^2 \tilde{\mathcal{T}})$ is written as

$$[s]_g = \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix} \in \begin{pmatrix} \tilde{\mathbb{E}}_a[0] \\ \tilde{\mathbb{E}}_{[a_0 a_1]}[2] \mid \tilde{\mathbb{E}}[1] \\ \tilde{\mathbb{E}}_a[2] \end{pmatrix}, \quad (19)$$

cf. [Ham09] for the conventions used here. In this form, the normal tractor connection is given by

$$\tilde{\nabla}_c^{\mathcal{AM}} \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix} = \begin{pmatrix} \tilde{D}_c \rho_a - \tilde{\mathbf{P}}_c^p \mu_{pa} - \tilde{\mathbf{P}}_{ca} \varphi \\ \tilde{D}_c \mu_{a_0 a_1} + 2\mathbf{g}_{c[a_0} \rho_{a_1]} + 2\tilde{\mathbf{P}}_{c[a_0} k_{a_1]} \mid \tilde{D}_c \varphi - \tilde{\mathbf{P}}_c^p k_p + \rho_c \\ \tilde{D}_c k_a - \mu_{ca} + \mathbf{g}_{ca} \varphi \end{pmatrix}.$$

In this notation, the curvature of the standard tractor connection $\tilde{\nabla}^{\tilde{\mathcal{T}}}$ is the section

$$\tilde{\Omega}_{c_0 c_1} = \begin{pmatrix} -\tilde{Y}_{ac_0 c_1} \\ \tilde{W}_{c_0 c_1 a_0 a_1} \mid 0 \\ 0 \end{pmatrix} \in \Gamma(\tilde{\mathbb{E}}_{[c_0 c_1]} \otimes \mathcal{AM}). \quad (20)$$

The BGG-splitting operator $L_0^{\mathcal{AM}} : \Gamma(\tilde{\mathbb{E}}^a) = \Gamma(\tilde{\mathbb{E}}_a[2]) \rightarrow \Gamma(\mathcal{AM})$ is given by

$$k_a \mapsto \begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix}, \text{ where}$$

$$\begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ k_a \end{pmatrix} = \begin{pmatrix} -\frac{1}{4n} \tilde{D}^p \tilde{D}_p k_a + \frac{1}{4n} \tilde{D}^p \tilde{D}_a k_p + \frac{1}{4n^2} \tilde{D}_a \tilde{D}^p k_p + \frac{1}{n} \tilde{P}_a^p k_p - \frac{1}{2n} \tilde{J} k_a \\ \tilde{D}_{[a_0} k_{a_1]} \mid -\frac{1}{2n} g^{pq} \tilde{D}_p k_q \\ k_a \end{pmatrix}. \quad (21)$$

It turns out that $\mathcal{H}_1^{\mathcal{AM}} \cong \tilde{\mathbb{E}}_{(ab)_0}[2]$, the trace-free part of $\tilde{\mathbb{E}}_{(ab)}[2]$, and the corresponding first BGG-operator is seen to be

$$\Theta_0^{\mathcal{AM}} : \Gamma(\tilde{\mathbb{E}}^a) = \Gamma(\tilde{\mathbb{E}}_a[2]) \rightarrow \Gamma(\tilde{\mathbb{E}}_{(ab)_0}[2]), \quad \Theta_0^{\mathcal{AM}}(\xi^a) = \tilde{D}_{(c}\xi_{a)0}. \quad (22)$$

Thus $\Theta_0^{\mathcal{AM}}$ is the conformal Killing operator and the solutions to the first BGG-equation are the conformal Killing fields. This condition may be equivalently stated as

$$\tilde{\nabla}_b^{\mathcal{AM}} s = \xi^a \tilde{\Omega}_{ab}, \quad (23)$$

where $s = L_0^{\mathcal{AM}}(\xi)$, see [Gov06, Čap08].

The standard pairing on \mathcal{AM} induced by the Killing form on $\tilde{\mathfrak{g}}$ is denoted as $\langle \cdot, \cdot \rangle : \mathcal{AM} \times \mathcal{AM} \rightarrow \mathbb{R}$.

3. FEFFERMAN-TYPE CONSTRUCTIONS

The construction of split-signature conformal structures from projective structures discussed in this article fits into a general scheme relating parabolic geometries of different types. Namely, it is an instance of the so called Fefferman-type construction, whose name and general procedure is motivated by Fefferman's construction of a canonical conformal structure induced by a CR structure, see [Čap06] and [ČS09] for a detailed discussion. In this section we recall the general principles, formulate a suitable condition for normality of the induced Cartan connection, and relate sections of associated tractor bundles.

3.1. General procedure. Suppose we have two pairs of semisimple Lie groups and parabolic subgroups, (G, P) and (\tilde{G}, \tilde{P}) , and a Lie group homomorphism $i : G \rightarrow \tilde{G}$ such that the derivative $i' : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ is injective. Assume further that the G -orbit of the origin in \tilde{G}/\tilde{P} is open and that the parabolic $P \subset G$ contains $Q := i^{-1}(\tilde{P})$, the preimage of $\tilde{P} \subset \tilde{G}$.

Given a parabolic geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) , one first forms the correspondence space

$$\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P P/Q, \quad (24)$$

which is also called the *Fefferman space*. Then $(\mathcal{G} \rightarrow \tilde{M}, \omega)$ is automatically a Cartan geometry of type (G, Q) . As a next step, one considers the extended bundle

$$\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P} \quad (25)$$

with respect to the homomorphism $Q \rightarrow \tilde{P}$. This is a principal bundle over \tilde{M} with structure group \tilde{P} and $j : \mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ denotes the natural inclusion.

The equivariant extension of $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ yields a unique Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ of type (\tilde{G}, \tilde{P}) such that

$$j^*\tilde{\omega} = i' \circ \omega. \quad (26)$$

Altogether, one obtains a functor from parabolic geometries $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) to parabolic geometries $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega})$ of type (\tilde{G}, \tilde{P}) .

The relation between the corresponding curvatures is as follows: The previous assumptions yield a linear isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$ and an obvious projection $\mathfrak{g}/\mathfrak{q} \rightarrow \mathfrak{g}/\mathfrak{p}$, where $\mathfrak{q} \subset \mathfrak{p}$ is the Lie algebra of $Q \subset P$. Composing these two maps one obtains a linear projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \rightarrow \mathfrak{g}/\mathfrak{p}$, whose dual map is denoted as $\varphi : (\mathfrak{g}/\mathfrak{p})^* \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$. Since $i' : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ is a homomorphism of Lie algebras, the curvature function $\tilde{\kappa} : \tilde{\mathcal{G}} \rightarrow \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ is related to $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ by

$$\tilde{\kappa} \circ j = (\Lambda^2 \varphi \otimes i') \circ \kappa. \quad (27)$$

We note that $\tilde{\kappa}$ is fully determined by this formula.

Since i' is an embedding, the notation is in most cases simplified so that we write $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$, etc. We proceed similarly on the level of Lie groups, provided that the map i is an embedding.

3.2. Normality. Starting with a regular, normal parabolic geometry of type (G, P) , it is not guaranteed that the parabolic geometry of type (\tilde{G}, \tilde{P}) produced by a Fefferman-type construction is regular and normal in general. For instance, it is known that the conformal parabolic geometries coming from partially integrable almost CR structures of hypersurface type are normal if and only if the almost CR structures are integrable, that is CR structures, see [ČG08]. In this section we formulate a sufficient condition, suitable for our purposes, that tells us when the parabolic geometry of type (\tilde{G}, \tilde{P}) coming from a normal parabolic of type (G, P) is itself normal.

Definition 3.1. *We call a Fefferman-type construction normal if it preserves normality, and we call it non-normal otherwise.*

The Kostant co-differential ∂^* , see (2), of a parabolic geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) can be explicitly expressed as follows. Let $X_1, \dots, X_n \in \mathfrak{g}$ be elements projecting to a basis of $\mathfrak{g}/\mathfrak{p}$, and let $Z_1, \dots, Z_n \in \mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ form the dual basis. Then, for any $\phi \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, $\partial^*\phi = \partial_1^*\phi + \partial_2^*\phi$, where

$$\partial_1^*\phi(X) := 2 \sum_{i=1}^n [\phi(X_i, X), Z_i], \quad \partial_2^*\phi(X) := \sum_{i=1}^n \phi(X_i, [Z_i, X]), \quad (28)$$

see [ČS09, Lemma 3.1.11]. Note that ∂_2^* vanishes identically for $|1|$ -graded Lie algebras, since in case of a $|1|$ -grading $[\tilde{\mathfrak{p}}_+, \tilde{\mathfrak{g}}] \subset \tilde{\mathfrak{p}}$.

Proposition 3.2. *Consider a Fefferman-type construction from parabolic geometries of type (G, P) to $|1|$ -graded parabolic geometries of type (\tilde{G}, \tilde{P}) corresponding to a Lie group homomorphism $i : G \rightarrow \tilde{G}$ between simple groups. Suppose further that the curvature function κ of the parabolic geometry (\mathcal{G}, ω) takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\mathfrak{g} \cap \tilde{\mathfrak{p}})$ and the two summands in the*

normality condition vanish separately, i.e. $\partial_1^* \circ \kappa = \partial_2^* \circ \kappa = 0$. Then the induced geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ is normal.

Proof. We use the Killing form \tilde{B} of $\tilde{\mathfrak{g}}$ (which restricts to the Killing form B of \mathfrak{g} up to a constant multiple) to identify $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ and $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{p}}_+$. Let $X_1, \dots, X_n \in \mathfrak{g}$ be elements projecting to a basis of $\mathfrak{g}/\mathfrak{p}$ and extend these by $X_{n+1}, \dots, X_m \in \mathfrak{p}$ such that X_1, \dots, X_m project to a basis of $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Let $\{Z_1, \dots, Z_n\}$, $Z_i \in \mathfrak{p}_+$, be the dual basis of $\{X_1 + \mathfrak{p}, \dots, X_n + \mathfrak{p}\}$ and let $\{\tilde{Z}_1, \dots, \tilde{Z}_m\}$, $\tilde{Z}_i \in \tilde{\mathfrak{p}}_+$ be the dual basis of $\{X_1 + \tilde{\mathfrak{p}}, \dots, X_m + \tilde{\mathfrak{p}}\}$. Then $\tilde{Z}_j - Z_j$, for $j = 1, \dots, n$, are contained in the orthogonal complement $\mathfrak{g}^\perp \subset \tilde{\mathfrak{g}}$ to \mathfrak{g} with respect to the Killing form: For $i = 1, \dots, n$, we have

$$\tilde{B}(X_i, \tilde{Z}_j - Z_j) = \tilde{B}(X_i, \tilde{Z}_j) - \tilde{B}(X_i, Z_j) = \delta_{i,j} - \delta_{i,j} = 0.$$

For $i = n+1, \dots, m$, we have $\tilde{B}(X_i, \tilde{Z}_j) = 0$ since $i \neq j$ and $\tilde{B}(X_i, Z_j) = 0$ since $X_i \in \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$. Finally, for $Y \in \mathfrak{q}$ we have $\tilde{B}(Y, \tilde{Z}_j) = 0$ since $\mathfrak{q} \subset \tilde{\mathfrak{p}}$ and $\tilde{Z}_j \in \tilde{\mathfrak{p}}_+$ and $\tilde{B}(Y, Z_j) = 0$ since $\mathfrak{q} \subset \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$.

Now suppose $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is the curvature function of (\mathcal{G}, ω) , and let $\tilde{\kappa} : \tilde{\mathcal{G}} \rightarrow \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ be the curvature function of the geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$. The latter geometry is normal if and only if

$$\tilde{\partial}^* \tilde{\kappa}(\tilde{u})(\tilde{X}) = 2 \sum_{i=1}^m [\tilde{\kappa}(\tilde{u})(X_i, \tilde{X}), \tilde{Z}_i] = 0 \quad (29)$$

for all $\tilde{u} \in \tilde{\mathcal{G}}$ and $\tilde{X} \in \tilde{\mathfrak{g}}$.

By construction, we know that $\tilde{\kappa}$ is a \tilde{P} -equivariant extension of κ and so elements of \mathfrak{p} insert trivially into $\tilde{\kappa}$. Since also $\tilde{\partial}^*$ is \tilde{P} -equivariant, to show normality of $\tilde{\kappa}$ it suffices to verify that

$$\tilde{\partial}^* \tilde{\kappa}(u)(X) = 2 \sum_{i=1}^n [\tilde{\kappa}(u)(X_i, X), \tilde{Z}_i] = 2 \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i] \quad (30)$$

vanishes for all $u \in \mathcal{G}$ and $X \in \mathfrak{g}$. Since by assumption

$$\partial_1^* \kappa(u)(X) = 2 \sum_{i=1}^n [\kappa(u)(X_i, X), Z_i] = 0,$$

we can rewrite $\tilde{\partial}^* \tilde{\kappa}(u)(X)$ as

$$2 \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i]. \quad (31)$$

We have observed that $\tilde{Z}_i - Z_i \in \mathfrak{g}^\perp$ and by construction $\kappa(u)(X_i, X) \in \mathfrak{g}$. Since the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^\perp$ is invariant under the action of \mathfrak{g} , this implies that $\tilde{\partial}^* \tilde{\kappa}(u)(X) = \sum_{i=1}^n [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i] \in \mathfrak{g}^\perp$. On the other hand, since by assumption $\kappa(u)(X_i, X) \in \tilde{\mathfrak{p}}$ and $\tilde{Z}_i \in \tilde{\mathfrak{p}}_+$, we have $\tilde{\partial}^* \tilde{\kappa}(u)(X) \in \tilde{\mathfrak{p}}_+$. But the intersection $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+$ is zero: Note that $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp$, so any element in $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+$ is orthogonal to $\mathfrak{g} + \tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}$. Since the Killing form is non-degenerate this implies $\mathfrak{g}^\perp \cap \tilde{\mathfrak{p}}_+ = 0$ and we conclude that $\tilde{\partial}^* \tilde{\kappa} = 0$. \square

3.3. Relation between sections of tractor bundles. The discussion here is analogous to the treatment in [ČG08]. The results in that reference also admit a straightforward generalisation to general Fefferman-type constructions for which P/Q is connected.

Suppose V is a \tilde{G} representation, which is then also a G representation, since $G \subset \tilde{G}$. Let $\mathcal{V} = \mathcal{G} \times_P V \rightarrow M$ and $\tilde{\mathcal{V}} = \tilde{\mathcal{G}} \times_{\tilde{P}} V = \mathcal{G} \times_Q V \rightarrow \tilde{M}$ be the associated tractor bundles. Let $\nabla^{\mathcal{V}}$ and $\nabla^{\tilde{\mathcal{V}}}$ be the tractor connections induced by ω and $\tilde{\omega}$, where $\tilde{\omega}$ is induced by ω via the Fefferman-type construction. Sections of \mathcal{V} bijectively correspond to P -equivariant functions $f : \mathcal{G} \rightarrow V$, while sections of $\tilde{\mathcal{V}}$ correspond to Q -equivariant functions $f : \mathcal{G} \rightarrow V$. In particular, since $Q \subset P$, every section of \mathcal{V} gives rise to a section of $\tilde{\mathcal{V}}$, and we can view $\Gamma(\mathcal{V}) \subset \Gamma(\tilde{\mathcal{V}})$.

Conversely, analogously to Proposition 3.3 in [ČG08] we have:

Proposition 3.3. *A section $s \in \Gamma(\tilde{\mathcal{V}})$ is contained in $\Gamma(\mathcal{V})$ (i.e. the corresponding Q -equivariant function f is indeed P -equivariant) if and only if $\nabla_{\xi}^{\tilde{\mathcal{V}}} s = 0$ for all ξ in the vertical bundle of $\tilde{M} \rightarrow M$.*

The idea is that since P/Q is connected, equivariance of f can be checked infinitesimally, which means that $\zeta_X(u) \cdot f = -X(f(u))$ for any $u \in \mathcal{G}$ and representative X of $\mathfrak{p}/\mathfrak{q}$, and this is equivalent to $\nabla_{\xi}^{\tilde{\mathcal{V}}} s = 0$ for all ξ in the vertical bundle $\mathcal{G} \times_Q \mathfrak{p}/\mathfrak{q}$.

The discussion in [ČG08] further shows that the tractor connection $\nabla^{\tilde{\mathcal{V}}}$ restricts to a connection on $\Gamma(\mathcal{V}) \subset \Gamma(\tilde{\mathcal{V}})$, which coincides with $\nabla^{\mathcal{V}}$. This implies a bijective correspondence between $\nabla^{\tilde{\mathcal{V}}}$ -parallel tractors in $\Gamma(\tilde{\mathcal{V}})$ and $\nabla^{\mathcal{V}}$ -parallel tractors in $\Gamma(\mathcal{V})$. If V is irreducible as a \tilde{G} -representation but has a G -invariant subspace $W \subset V$, then this correspondence restricts to a bijective correspondence between parallel sections of $\tilde{\mathcal{W}} = \mathcal{G} \times_Q W \rightarrow \tilde{M}$ and parallel sections of $\mathcal{W} = \mathcal{G} \times_P W \rightarrow M$.

4. FROM PROJECTIVE TO CONFORMAL STRUCTURES OF SPLIT-SIGNATURE

The following section discusses in detail the construction from projective to split-signature conformal structures as a Fefferman-type construction. Firstly, we specify the algebraic setting. Secondly, we obtain a geometric description of the Fefferman space, and we show that our construction does not preserve normality unless we start with a 2-dimensional projective structure. Thirdly, we discuss the notion of reduced Weyl structures and reduced scales on the Fefferman space. Fourthly, as an aside, the Fefferman construction of split-signature conformal structures coming from torsion-free Lagrangean contact structures is discussed.

4.1. Algebraic part. Here we specify the general setup for Fefferman-type constructions from subsection 3.1 according to the description of oriented projective and conformal spin structures given in subsection 2.2 and 2.3, respectively. Let $\mathbb{R}^{n+1, n+1}$ be the real vector space \mathbb{R}^{2n+2} with an inner product, h , of split-signature. Let $\Delta_+^{n+1, n+1}$ and $\Delta_-^{n+1, n+1}$ be the irreducible

spin representations of

$$\tilde{G} := \text{Spin}(n+1, n+1)$$

as in 2.3.2. We fix two pure spinors $s_F \in \Delta_-^{n+1, n+1}$ and $s_E \in \Delta_\pm^{n+1, n+1}$ with non-trivial pairing, which is assigned for later use to be

$$\langle s_E, s_F \rangle = -\frac{1}{2}.$$

Note that s_E lies in $\Delta_+^{n+1, n+1}$ if n is even or in $\Delta_-^{n+1, n+1}$ if n is odd.

Let us denote by $E, F \subset \mathbb{R}^{n+1, n+1}$ the kernels of s_E, s_F with respect to the Clifford multiplication, i.e.

$$\begin{aligned} E &:= \{X \in \mathbb{R}^{n+1, n+1} : X \cdot s_E = 0\}, \\ F &:= \{X \in \mathbb{R}^{n+1, n+1} : X \cdot s_F = 0\}. \end{aligned}$$

The purity of s_E and s_F means that E and F are maximally isotropic subspaces in $\mathbb{R}^{n+1, n+1}$. The other assumptions guarantee that E and F complementary and dual each other via the inner product h . Hence we use the decomposition

$$\mathbb{R}^{n+1, n+1} = E \oplus F \cong \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1*} \quad (32)$$

to identify the spinor representation $\Delta = \Delta_+^{n+1, n+1} \oplus \Delta_-^{n+1, n+1}$ with the exterior power algebra $\Lambda^\bullet E \cong \Lambda^\bullet \mathbb{R}^{n+1}$, whose irreducible subrepresentations are $\Delta_-^{n+1, n+1} \cong \Lambda^{\text{even}} \mathbb{R}^{n+1}$ and $\Delta_+^{n+1, n+1} \cong \Lambda^{\text{odd}} \mathbb{R}^{n+1}$. When n is even, respectively, odd, we can identify $(\Delta_-^{n+1, n+1})^* \cong \Delta_+^{n+1, n+1}$, respectively $(\Delta_+^{n+1, n+1})^* \cong \Delta_-^{n+1, n+1}$.

Now, let us consider the subgroup in \tilde{G} defined by

$$G := \{g \in \text{Spin}(n+1, n+1) : g \cdot s_E = s_E, g \cdot s_F = s_F\}.$$

It preserves the decomposition (32), so that the restriction of the action to F is dual to the restriction to E , and the volume form is determined by s_E and s_F . Hence G is isomorphic to $\text{SL}(n+1)$ and this defines an embedding $i : G \hookrightarrow \tilde{G}$.

The G -invariant decomposition (32) determines a G -invariant skew-symmetric involution $K \in \mathfrak{so}(n+1, n+1)$ acting by the identity on E and minus the identity on F . The relationship among K , s_E and s_F may be expressed as

$$h(X, K(Y)) = -h(K(X), Y) = 2 \langle s_E, (X \wedge Y) \cdot s_F \rangle, \quad (33)$$

where

$$\begin{aligned} (X \wedge Y) \cdot s_F &= \frac{1}{2} (X \cdot Y \cdot s_F - Y \cdot X \cdot s_F) \\ &= X \cdot Y \cdot s_F + h(X, Y) s_F. \end{aligned}$$

Moreover, it follows that Clifford action of K on the spinors is

$$K \cdot s_F = (2n+2)s_F, \quad K \cdot s_E = -(2n+2)s_E.$$

Here we identify $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1, n+1)$, the Lie algebra of \tilde{G} , with $\Lambda^2 \mathbb{R}^{n+1, n+1}$. The spin action of $\tilde{\mathfrak{g}}$ is denoted by \bullet so that $A \bullet s = -\frac{1}{4} A \cdot s$, for any $A \in \tilde{\mathfrak{g}}$

and $s \in \Delta$. In particular,

$$K \bullet s_F = -\frac{1}{2}(n+1)s_F, \quad K \bullet s_E = \frac{1}{2}(n+1)s_E.$$

It is convenient to split $\tilde{\mathfrak{g}}$ in terms of irreducible \mathfrak{g} -modules as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \underbrace{(E \otimes F)_0}_{\mathfrak{g}} \oplus \underbrace{(E \otimes F)_{Tr} \oplus \Lambda^2 E \oplus \Lambda^2 F}_{\mathfrak{g}^\perp}, \quad (34)$$

where $(E \otimes F)_{Tr} = \mathbb{R}K$, and K acts as

$$[K, \phi] = 2\phi, \quad [K, \psi] = -2\psi, \quad [K, \lambda] = 0, \quad (35)$$

for any $\phi \in \Lambda^2 E$, $\psi \in \Lambda^2 F$ and $\lambda \in E \otimes F$. Further, the annihilators of s_E and s_F in $\tilde{\mathfrak{g}}$ are the subalgebras

$$\begin{aligned} \ker s_E &:= \{\phi \in \tilde{\mathfrak{g}} : \phi \bullet s_E = 0\} = (E \otimes F)_0 \oplus \Lambda^2 E, \\ \ker s_F &:= \{\phi \in \tilde{\mathfrak{g}} : \phi \bullet s_F = 0\} = (E \otimes F)_0 \oplus \Lambda^2 F. \end{aligned} \quad (36)$$

4.1.1. Homogeneous model. The homogeneous model for conformal spin structures of split-signature (n, n) , $\tilde{G}/\tilde{P} \cong S^n \times S^n$, is the space of isotropic rays in $\mathbb{R}^{n+1, n+1}$. The subgroup $G \subset \tilde{G}$ does not act transitively on that space — according to the decomposition (32), there are three orbits: the set of rays contained in E , the set of rays contained in F , and the set of isotropic rays that are neither contained in E nor in F . Note that only the last orbit is open in \tilde{G}/\tilde{P} , which is one of the requirements from 3.1.

Therefore, we define $\tilde{P} \subset \tilde{G}$ to be the stabiliser of a ray through a light-like vector

$$\tilde{v} \in \mathbb{R}^{n+1, n+1} \setminus (E \cup F).$$

Denoting by $Q = i^{-1}(\tilde{P})$ the stabiliser of the ray $\mathbb{R}_+ \tilde{v}$ in G , we have got the identification of G/Q with the open orbit of the origin in \tilde{G}/\tilde{P} . The subgroup Q , which is not parabolic, is contained in the parabolic subgroup $P \subset G$ defined as the stabiliser in G of the ray through the projection of \tilde{v} to E ,

$$v = (\tilde{v})_E.$$

In particular, G/P is the standard projective sphere S^n , the homogeneous model of oriented projective structures of dimension n , and $G/Q \rightarrow G/P$ is the canonical fibration with the standard fibre P/Q , whose total space is the model Fefferman space.

Let us denote by $L = \mathbb{R}\tilde{v}$ the line spanned by the light-like vector \tilde{v} and let L^\perp be the orthogonal complement in $\mathbb{R}^{n+1, n+1}$ with respect to h . The tangent space of G/Q in the origin can be seen in three different ways, namely,

$$(L^\perp/L)[1] \cong \mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

The latter isomorphism is induced by the embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, the former one by the standard action of $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ on the vector $\tilde{v} \in \mathbb{R}^{n+1, n+1}$. The particular weight in the first term is chosen so that both these identifications are Q -equivariant.

There are several natural Q -invariant objects that in turn yield distinguished geometric objects on the general Fefferman space. The n -dimensional Q -invariant subspace

$$f := (\bar{F}/L)[1] \subset (L^\perp/L)[1] \quad \text{where} \quad \bar{F} := F \cap L^\perp,$$

which is, under one of the previous identifications,

$$f \cong \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q},$$

the kernel of the projection $\mathfrak{g}/\mathfrak{q} \rightarrow \mathfrak{g}/\mathfrak{p}$. Another n -dimensional Q -invariant subspace is

$$e := (\bar{E}/L)[1] \subset (L^\perp/L)[1] \quad \text{where} \quad \bar{E} := E \cap L^\perp.$$

The intersection $e \cap f$ is 1-dimensional with a distinguished Q -invariant generator that corresponds to the G -invariant involution $K \in \tilde{\mathfrak{g}}$,

$$k := K + \tilde{\mathfrak{p}} \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

Note that all these objects are isotropic with respect to the natural conformal class induced by the restriction of h to $L^\perp \subset \mathbb{R}^{n+1,n+1}$. In particular, both e and f are maximally isotropic subspaces such that

$$k \in e \cap f \subset k^\perp = e + f. \quad (37)$$

4.2. The Fefferman space and the induced structure. The pairs of Lie groups (G, P) and (\tilde{G}, \tilde{P}) from the previous subsection satisfy all the properties from 3.1 to launch the Fefferman-type construction. At this place we can formulate first immediate properties of the induced conformal structure. Foremost, we can describe the Fefferman space \tilde{M} in terms of the initial projective manifold M .

Proposition 4.1. *The Fefferman-type construction for the pairs of Lie groups (G, P) and (\tilde{G}, \tilde{P}) as described in subsection 4.1 yields a natural construction of conformal spin structures (\tilde{M}, \mathbf{c}) of signature (n, n) from n -dimensional oriented projective structures (M, \mathbf{p}) . The Fefferman space \tilde{M} is identified with the total space of the weighted cotangent bundle without the zero section $T^*M(2) \setminus \{0\}$.*

Proof. The first part of the statement is obvious from the general setting for Fefferman-type constructions, cf. subsection 3.1, and the Cartan-geometric description of oriented projective and conformal spin structures, cf. subsection 2.2 and 2.3, respectively.

The second part is shown due to two natural identifications: On the one hand, the Fefferman space is by (24) equal to the total space of the associated bundle $\tilde{M} \cong \mathcal{G} \times_P P/Q$ over M . On the other hand, the weighted cotangent bundle to M is identified with the associated bundle $T^*M(2) \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^*(2)$ with respect to action of P induced by the adjoint action and the representation (12) for $w = 2$. Hence it remains to verify that $(\mathfrak{g}/\mathfrak{p})^*(2) \setminus \{0\} \cong \mathfrak{p}_+(2)$ is the homogeneous space of P/Q , i.e. that the action of P is transitive and Q is a stabiliser of a non-zero element. But this is a purely algebraic task, which may be easily checked in a concrete matrix realisation. (According to the choices made in B.1, one observes

that Q is the stabiliser of an element from (72) corresponding to $U = 0$ and $w = 1$.) \square

The induced conformal structure on \widetilde{M} has a lot of specific features. In the following statement, the previous algebraic setup is translated into the actual geometric terms.

Proposition 4.2. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal spin structure induced from an oriented projective structure (M, \mathbf{p}) by the Fefferman-type construction. Let \widetilde{T} , $\widetilde{\mathcal{A}M}$ and $\widetilde{\mathcal{S}}_{\pm}$ be the conformal standard, adjoint and spinor tractor bundle, respectively. Then \widetilde{M} carries the following tractor fields, which are all parallel with respect to the induced tractor connections $\widetilde{\nabla}^{ind}$ on the respective bundles:*

- (a) *pure tractor spinors $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_{\pm})$ and $\mathbf{s}_F \in \Gamma(\widetilde{\mathcal{S}}_{-})$ with non-trivial pairing,*
- (b) *an involution $\mathbf{K} \in \Gamma(\widetilde{\mathcal{A}M})$ acting by the identity on $\widetilde{\mathcal{E}} \subset \widetilde{T}$, the kernel of \mathbf{s}_E , and by minus the identity on $\widetilde{\mathcal{F}} \subset \widetilde{T}$, the kernel of \mathbf{s}_F ,*

In particular, the standard tractor bundle decomposes into two maximally isotropic subbundles $\widetilde{T} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$ that are also parallel with respect to $\widetilde{\nabla}^{ind}$.

The tractor fields \mathbf{s}_E , \mathbf{s}_F , respectively \mathbf{K} , descend to pure spinors $\eta \in \Gamma(\widetilde{\mathcal{S}}_{\pm}[\frac{1}{2}])$, $\chi \in \Gamma(\widetilde{\mathcal{S}}_{-}[\frac{1}{2}])$, respectively a light-like vector field $k \in \Gamma(T\widetilde{M})$, which are all nowhere-vanishing and satisfy:

- (c) *the kernel $\widetilde{f} := \ker \chi$ coincides with $V\widetilde{M} \subset T\widetilde{M}$, the vertical subbundle of the projection $\widetilde{M} \rightarrow M$,*
- (d) *the kernel $\widetilde{e} := \ker \eta$ intersects $V\widetilde{M}$ in a 1-dimensional subspace,*
- (e) *the vector field k belongs to that intersection.*

Proof. The G -invariant objects s_E , s_F and K introduced in subsection 4.1 induce the tractor fields \mathbf{s}_E , \mathbf{s}_F and \mathbf{K} so that $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_{\pm} = \mathcal{G} \times_P \Delta_{\pm})$ corresponds to the constant (P -equivariant) map $\mathcal{G} \rightarrow \Delta_{\pm}$ and similarly for others. The purity and non-trivial pairing of \mathbf{s}_E and \mathbf{s}_F follows from the purity and non-trivial pairing of s_E and s_F . The kernels of \mathbf{s}_E and \mathbf{s}_F correspond to the kernels of s_E and s_F so that $\widetilde{\mathcal{E}} = \widetilde{\mathcal{G}} \times_{\widetilde{P}} E$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{G}} \times_{\widetilde{P}} F$, respectively, and the decomposition $\widetilde{T} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$ corresponds to the G -invariant decomposition (32). Hence all these tractor objects are automatically parallel with respect to the induced tractor connections. Let $\eta = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_E)$, $\chi = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_F)$ and $k = \Pi_0^{\widetilde{\mathcal{A}M}}(\mathbf{K})$ be the corresponding underlying objects on \widetilde{M} .

The filtration $L \subset L^{\perp} \subset \mathbb{R}^{n+1, n+1}$ from 4.1.1 gives rise to the filtration of the standard tractor bundle, which in terms of (14) looks like

$$\begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ 0 \\ 0 \end{pmatrix} \subset \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ 0 \end{pmatrix} \subset \widetilde{T}. \quad (38)$$

In particular, the subspaces $\bar{E}, \bar{F} \subset L^{\perp}$ are distinguished by the middle slot. The Q -invariant maximally isotropic subspaces $e, f \subset \mathfrak{g}/\mathfrak{q}$ determine the distributions in $T\widetilde{M}$, namely, $\mathcal{G} \times_Q e$ and $\mathcal{G} \times_Q f$. Now, according to the

tractor Clifford action (17), it follows that these are precisely the kernels of the spinors η and χ . Since these subspaces are maximally isotropic, the corresponding spinors are pure. Since $f \cong \mathfrak{p}/\mathfrak{q}$ is the kernel of the projection $\mathfrak{g}/\mathfrak{q} \rightarrow \mathfrak{g}/\mathfrak{p}$, the corresponding subbundle \tilde{f} is identified with the vertical subbundle of the projection $\tilde{M} \rightarrow M$. Finally, the intersection $e \cap f$ is 1-dimensional and it is generated by the projection of $K \in \tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Hence the corresponding vector field k on \tilde{M} belongs to the intersection $\tilde{e} \cap \tilde{f}$.

It remains to show that the spinors η , χ and the vector field k are nowhere-vanishing. Since the skew-symmetric involution K acts by the identity on E and minus the identity on F , it cannot be contained in $\tilde{\mathfrak{p}}$ (this would imply that it preserves the line $L = \mathbb{R}\tilde{v}$, which is impossible as \tilde{v} is neither contained in E nor in F). Hence k is nowhere-vanishing. But then also the underlying spinors are nowhere-vanishing. \square

Now we show that, in the non-flat case, our Fefferman-type construction is normal if and only if $\dim M = 2$.

Proposition 4.3. *Let $(\mathcal{G} \rightarrow M, \omega)$ be a normal projective parabolic geometry and let $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega}^{ind})$ be the conformal parabolic geometry obtained by the Fefferman-type construction described in 4.1.*

- (a) *If $\dim M = 2$ then $\tilde{\omega}^{ind}$ is normal.*
- (b) *If $\dim M > 2$ then $\tilde{\omega}^{ind}$ is normal if and only if ω is flat.*

Moreover, independently of the dimension of M , $\tilde{\omega}^{ind}$ is flat if and only if ω is flat.

Proof. It is a general feature of Fefferman-type constructions that $\tilde{\omega}^{ind}$ is flat if and only if ω is flat. This immediately follows from the relation between the respective curvatures that is expressed in (27).

(a) In the special case of a projective structure in dimension $n = 2$ the curvature function of a normal projective Cartan connection takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}_+$, see e.g. [ČS09]. It is easily seen from the explicit matrices that $\mathfrak{p}_+ \subset \tilde{\mathfrak{p}} \cap \mathfrak{g}$. We can thus apply Proposition 3.2 in this case, which proves the statement.

(b) If ω is flat, then $\tilde{\omega}^{ind}$ is also flat, in particular, it is normal. If $\tilde{\omega}^{ind}$ is normal, then it is torsion-free, i.e. the curvature function $\tilde{\kappa}$ takes values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes (\tilde{\mathfrak{p}} \cap \mathfrak{g})$. But this is only possible if the harmonic curvature of the original projective geometry takes values in a P -submodule of $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}/\mathfrak{p}_+$ that is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\tilde{\mathfrak{p}} \cap \mathfrak{g})/\mathfrak{p}_+$, and there is no such non-trivial P -invariant subspace. \square

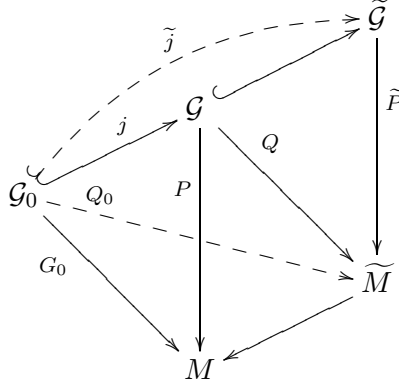
The construction for the lowest admissible dimension $n = 2$ is indeed specific, hence the geometric conclusions and characterisations are relatively easy to gain, cf. subsection 4.5. On the other hand, the reasoning for general dimension $n > 2$ is much more subtle and it occupies most of the paper starting with section 5.

4.3. Reduced Weyl structures and reduced scales. As a technical preliminary for further study we now relate the Weyl structures of the original

projective Cartan geometry (\mathcal{G}, ω) on M and the induced conformal geometry $(\tilde{\mathcal{G}}, \tilde{\omega}^{ind})$ on the Fefferman space \tilde{M} . The former ones, for short projective Weyl structures, are just reductions $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ of the principal P -bundle over M to the structure group G_0 . The latter ones, for short conformal Weyl structures, are reductions $\tilde{\mathcal{G}}_0 \xrightarrow{\tilde{j}} \tilde{\mathcal{G}}$ of the \tilde{P} -bundle over \tilde{M} to the group \tilde{G}_0 . Further reductions to the semisimple part of G_0 and \tilde{G}_0 , respectively, yield the exact Weyl structures, cf. paragraph 2.1.3.

Proposition 4.4. *Any projective (exact) Weyl structure on M induces a conformal (exact) Weyl structure on the Fefferman space \tilde{M} . In particular, any projective scale induces a conformal scale.*

Proof. Fix a Levi subgroup $G_0 \subset P$, such that $G_0 \cong P/P_+$. Let Q_0 be the intersection of G_0 with $Q = G \cap \tilde{P}$. Since $P_+ \subset Q$, $Q_0 \cong Q/P_+$, and thus $G_0/Q_0 \cong P/Q$. Therefore a reduction $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ from P to G_0 over the manifold M induces a reduction from Q to Q_0 over \tilde{M} . Composing the embedding $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ with the natural embedding $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}} = \mathcal{G} \times_Q \tilde{P}$ over \tilde{M} , one obtains a reduction $\mathcal{G}_0 \xrightarrow{\tilde{j}} \tilde{\mathcal{G}}$ from \tilde{P} to Q_0 over \tilde{M} .



We can choose a Levi subgroup $\tilde{G}_0 \subset \tilde{P}$, $\tilde{G}_0 \cong \tilde{P}/\tilde{P}_+$, that contains Q_0 , see Appendix B.1 for an explicit realisation. Having done that, we define $\tilde{\mathcal{G}}_0 := \mathcal{G}_0 \times_{Q_0} \tilde{G}_0$, a \tilde{G}_0 -principal bundle over \tilde{M} . Then \mathcal{G}_0 is naturally embedded into $\tilde{\mathcal{G}}_0$ and the above embedding \tilde{j} canonically extends to an embedding of the \tilde{G}_0 -bundle $\tilde{\mathcal{G}}_0$ into the \tilde{P} -bundle $\tilde{\mathcal{G}}$ over \tilde{M} , which we simply denote by \tilde{j} again. Altogether, for a projective Weyl structure $\mathcal{G}_0 \xrightarrow{j} \mathcal{G}$ we have constructed a conformal Weyl structure $\tilde{\mathcal{G}}_0 \xrightarrow{\tilde{j}} \tilde{\mathcal{G}}$.

Similar arguments hold true also for exact Weyl structures, i.e. the reductions of the respective bundles to the subgroups $G_0^{ss} \subset P$ and $\tilde{G}_0^{ss} \subset \tilde{P}$, the semisimple parts of G_0 and \tilde{G}_0 . (According to the previous choices, the subgroup \tilde{G}_0^{ss} already contains $G_0^{ss} \cap Q$.) \square

A version of the above result in a more general context was proved in [Alt10].

Definition 4.5. *Conformal Weyl structures induced by projective Weyl structures as above will be called reduced Weyl structures. Similar terminology applies also to exact Weyl structures; corresponding conformal scales will be called reduced scales.*

Remark 4.6. Recall that the pullback of a Cartan connection with respect to a Weyl structure determines a connection, called Weyl connection, and a T^*M -valued one-form, called Schouten tensor, see section 2.1.3. Let us remark for later use that the Weyl connection \tilde{D}^{ind} and Schouten tensor \tilde{P}^{ind} for the induced Cartan connection $\tilde{\omega}^{ind}$ have special properties in any reduced Weyl structure: First, if \tilde{j} is a reduced Weyl structure, then the structure group of $\tilde{\mathcal{G}}_0$ is further reduced to Q_0 and the $\tilde{\mathfrak{g}}_0$ -part of $\tilde{j}^*\tilde{\omega}^{ind}$ defines a Q_0 -connection. This implies that the corresponding induced Weyl connection \tilde{D}^{ind} on any associated bundle $\mathcal{G}_0 \times_{Q_0} V$ preserves all subbundles corresponding to Q_0 submodules of V . Second, it is immediately seen from the fact that the projection of \mathfrak{g} onto $\tilde{\mathfrak{p}}_+$ equals $f[-2]$ that the $\tilde{\mathfrak{p}}_+$ -part of $\tilde{j}^*\tilde{\omega}^{ind}$ takes values in $f[-2]$, and hence the Schouten tensor \tilde{P}^{ind} is a one-form on \tilde{M} with values in the corresponding bundle $\tilde{f}[-2] \subset T^*\tilde{M}$.

In section 2.2 and 2.3 we have fixed suitable parametrisations of density bundles on projective and conformal manifolds, which are denoted by $\mathbb{E}(w) \rightarrow M$ and $\tilde{\mathbb{E}}[w] \rightarrow \tilde{M}$, respectively. Each of these bundles may as well be described as an associated bundle to the bundle of scales (only the central part of the reductive subgroup G_0 , respectively \tilde{G}_0 , acts nontrivially). Hence everywhere positive sections of each of these bundles may as well be considered as scales; in the following we do not really distinguish between these two concepts. On that account, we need to understand the relation between the weights of projective and conformal densities for our Fefferman-type construction:

Lemma 4.7. *Any projective w -density on M induces a conformal w -density on the Fefferman space \tilde{M} .*

Proof. The projective and conformal density bundle is defined via the representation of P and \tilde{P} as in (12) and (13), respectively. In order to compare the corresponding weights in the case of the Fefferman construction, it suffices to restrict the representations to Q , which is a subgroup both in P and \tilde{P} (in accord with the embedding $i : G \hookrightarrow \tilde{G}$). A direct check reveals the statement. \square

An intrinsic characterisation of reduced scales among all conformal ones is formulated in Proposition 6.2.

4.4. The intermediate Lagrangean contact structure. We now briefly discuss a natural intermediate step to the construction of Section 4.2, namely, the canonical Lagrangean contact structure on the projectivised cotangent bundle $\mathcal{P}(T^*M)$ of a projective manifold M .

A *Lagrangean contact structure* on M' consists of a contact distribution $\mathcal{H} \subset TM'$ together with a decomposition $\mathcal{H} = e' \oplus f'$ into two subbundles that are maximally isotropic with respect to the Levi form $\mathcal{H} \times \mathcal{H} \rightarrow TM'/\mathcal{H}$.

Such structure is equivalently encoded as a normal parabolic geometry of type (G, P') , where $G = \mathrm{SL}(n+1)$, $\dim M' = 2n-1$, and $P' \subset G$ is the stabiliser of a flag of type line–hyperplane in the standard representation \mathbb{R}^{n+1} . For suitable choices as in B.1, the Lie algebra to P' consists of matrices of the form

$$\mathfrak{p}' = \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & 0 & c \end{pmatrix} \subset \mathfrak{p}, \quad (39)$$

where $\mathfrak{p} \subset \mathfrak{g}$, the Lie algebra of $P \subset G$, appertains to projective structures as before. Given a projective Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$, it turns out the correspondence space $M' := \mathcal{G}/P'$ can be identified with $\mathcal{P}(T^*M)$ and the Cartan geometry $(\mathcal{G} \rightarrow M', \omega)$ of type (G, P') covers the natural Lagrangean contact structure. See e.g. [Čap05, ČS09] for details.

The previous construction was based on the inclusion $P' \subset P$. On the other hand, there is an inclusion $Q \subset P'$, where $Q = i^{-1}(\tilde{P})$, $\tilde{P} \subset \tilde{G} = \mathrm{Spin}(n+1, n+1)$, and $i : G \rightarrow \tilde{G}$ as before. This allows us to consider a Fefferman-type construction for the pairs (G, P') and (\tilde{G}, \tilde{P}) , which induces a conformal spin structure on $\tilde{M} = \mathcal{G}/Q$ to a Lagrangean contact structure on M' . This construction is indeed very similar to the original Fefferman construction; one deals with different real forms of the same complex Lie groups in the two cases. That is why the following statements and their proofs are analogous to those in the literature for the CR case, cf. e.g. [Gra87, ČG10, Lei08].

Concluding, the construction we are interested in for the purposes this article can be regarded as the composition of the correspondence space construction from a projective structure on M to a Lagrangean contact structure on $M' = \mathcal{P}(T^*M)$ with the Fefferman-type construction to a conformal spin structure on $\tilde{M} = T^*M(2) \setminus \{0\}$. The induced conformal objects from Proposition 4.2 correspond to the induced objects on M' , so that the vertical subbundle of the projection $\tilde{M} \rightarrow M'$ is spanned by k and the decomposition $k^\perp = \tilde{e} \oplus \tilde{f} \subset T\tilde{M}$ descends to the decomposition $\mathcal{H} = e' \oplus f' \subset TM'$ (cf. (37) and the respective matrix realisations).

Proposition 4.8. *Let $(\mathcal{G} \rightarrow M', \omega)$ be the normal parabolic geometry encoding a Lagrangean contact structure and let $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega})$ be the conformal parabolic geometry obtained by the Fefferman-type construction described in the previous paragraph. Then $\tilde{\omega}$ is normal if and only if ω is torsion-free.*

Proof. Let ω be torsion-free and let κ be the corresponding curvature function. It is easy to show that the two summands in the normality condition (28) vanish separately on κ and that κ takes values in the P' -submodule $\Lambda^2(\mathfrak{g}/\mathfrak{p}')^* \otimes (\mathfrak{g}_0'^{ss} \oplus \mathfrak{p}'_+) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p}')^* \otimes \mathfrak{p}'$, see e.g. subsection 3.8 in [ČŽ09]. Since $\mathfrak{g}_0'^{ss} \oplus \mathfrak{p}'_+ \subset \mathfrak{q}$ and $\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}}$, all assumptions of Proposition 3.2 are satisfied, hence the normality of $\tilde{\omega}$ follows.

The converse direction is obvious since normal $\tilde{\omega}$ is torsion-free and concurrently $\mathfrak{g} \cap \tilde{\mathfrak{p}} = \mathfrak{q}$ is contained in $\mathfrak{p}' \subset \mathfrak{g}$. Hence κ takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p}')^* \otimes \mathfrak{p}'$, i.e. ω is torsion-free. \square

Remark 4.9. To relate Proposition 4.8 to Proposition 4.3, we remark that in dimension three any Lagrangean contact structure is automatically torsion-free, while in higher dimensions the Lagrangean contact structure induced by a projective structure as above is torsion-free if and only if it is flat, or equivalently, if and only if the initial projective structure is flat (see e.g. Proposition 4.4.2. in [ČS09]). Note that the torsion-freeness of general Lagrange contact structure is equivalent to the integrability of both distributions $e', f' \subset TM'$.

The aim of the rest of the section is to derive a local characterisation of split-signature conformal structures arising from torsion-free Lagrangean contact structures. As Remark 4.9 shows this is a small detour from the main objectives of this article, but straightforward to obtain and interesting in its own right.

Proposition 4.10. *Let (M, \mathbf{c}) be a conformal spin manifold of split-signature (n, n) . Then the following conditions are locally equivalent:*

- (a) *The spin tractor bundle admits two pure parallel spin tractors $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_+)$ and $\mathbf{s}_F \in \Gamma(\widetilde{\mathcal{S}}_-)$ with non-trivial pairing.*
- (b) *The conformal holonomy $\text{Hol}(\mathbf{c})$ reduces to $\text{SL}(n+1) \subset \text{Spin}(n+1, n+1)$ preserving a decomposition into maximally isotropic subspaces $E \oplus F = \mathbb{R}^{(n+1, n+1)}$.*
- (c) *The adjoint tractor bundle admits a parallel involution, i.e. $\mathbf{K} \in \Gamma(\widetilde{\mathcal{AM}})$ such that $\mathbf{K}^2 = 1$ and $\widetilde{\nabla}^{nor} \mathbf{K} = 0$.*

Proof. Given two parallel pure tractor spinors $\mathbf{s}_E, \mathbf{s}_F$ with non-trivial pairing, the conformal holonomy $\text{Hol}(\mathbf{c})$ is contained in $\text{SL}(n+1) \subset \text{Spin}(n+1, n+1)$, which preserves the decomposition into the kernels $E = \ker(s_E)$ and $F = \ker(s_F)$, respectively. Thus (a) implies (b). Given such a holonomy invariant decomposition, we obtain a skew-symmetric involution K by defining it as the identity on E and minus the identity on F , and this determines a parallel adjoint tractor \mathbf{K} . Thus (b) implies (c).

To verify that, locally, (c) implies (a), first notice that the skew symmetry of \mathbf{K} implies that the eigenspaces to $+1$ and -1 have the same dimension and are dual each other with respect to the tractor metric; thus the conformal holonomy $\text{Hol}(\mathbf{c})$ is reduced to $\text{GL}(n+1)$. Next, we can use a result of Proposition 2.1 in [ČG10] which shows that if $s \in \Gamma(\widetilde{\mathcal{AM}})$ is a parallel adjoint tractor, then the pairing with the normal conformal curvature vanishes. In particular for $s = \mathbf{K}$, $\langle \widetilde{\Omega}^{nor}, \mathbf{K} \rangle = 0$, which means that the curvature has values in $\mathfrak{sl}(n+1)$. Since $\mathfrak{sl}(n+1) \subset \mathfrak{gl}(n+1)$ is an ideal, this implies that the conformal holonomy Lie algebra is contained in $\mathfrak{sl}(n+1)$. Thus there are, locally, parallel pure tractor spinors \mathbf{s}_E and \mathbf{s}_F . (See e.g. [Lei08] for a similar argument in the CR-case.) \square

Corollary 4.11. *Let $(\widetilde{M}, \mathbf{c})$ be the induced conformal spin structure on the Fefferman space over a torsion-free Lagrangean contact manifold. Then \widetilde{M} carries tractor fields $\mathbf{s}_E, \mathbf{s}_F$, and \mathbf{K} satisfying the properties from Proposition 4.10. The underlying objects are pure twistor spinors η, χ , and a light-like conformal Killing field k , which are all nowhere-vanishing.*

Proof. Completely analogously to the proof of Proposition 4.2 one shows that the Fefferman space over a Lagrangean contact manifold carries induced tractor objects \mathbf{s}_E , \mathbf{s}_F , and \mathbf{K} having the required algebraic properties and non-vanishing projecting slots. By construction, they are parallel with respect to the induced tractor connection $\tilde{\nabla}^{ind}$ and, by Proposition 4.8, we know that $\tilde{\nabla}^{nor} = \tilde{\nabla}^{ind}$. Thus they are parallel for the normal tractor connection and the corresponding underlying objects are pure twistor spinors and a light-like conformal Killing field, respectively. \square

Applying results from [ČGH14] and [Čap05] easily yields a converse to Corollary 4.11. Suppose we are given a $2n$ -dimensional split-signature conformal structure (\tilde{M}, \mathbf{c}) and a holonomy reduction to $\mathrm{SL}(n+1) \subset \mathrm{Spin}(n+1, n+1)$ determined by two parallel pure tractor spinors $\mathbf{s}_E \in \Gamma(\tilde{\mathcal{S}}_+)$ and $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-)$. Let χ, η be the underlying twistor spinors and k be the underlying conformal Killing field. Recall from Section 4.1.1 that \tilde{G}/\tilde{P} decomposes under G into three orbits: an open one and two closed n -dimensional ones. Theorem in [ČGH14] shows that, correspondingly, we have a decomposition of \tilde{M} into so-called curved orbits and, provided they are non-empty, each of them carries an induced Cartan geometry of the same type as the corresponding orbit in the homogeneous model. Here this means that the closed n -dimensional curved orbits carry induced Cartan geometries of type $(\mathrm{SL}(n+1), P)$ and thus inherit projective structures. The open curved orbit carries an induced Cartan geometry of type $(\mathrm{SL}(n+1), Q)$. In terms of underlying data the decomposition of \tilde{M} can be described as the decomposition into the zero sets of χ and η , respectively, and the open subset where both spinors, and thus k , are non-vanishing.

Proposition 4.12. *Let (\tilde{M}, \mathbf{c}) be a conformal spin manifold of dimension $2n$ satisfying the conditions from Proposition 4.10 and assume that the underlying twistor spinors η, χ and the conformal Killing field k are nowhere-vanishing. Then, around every point, (\tilde{M}, \mathbf{c}) is locally equivalent to the induced conformal structure on the Fefferman space over a torsion-free Lagrangean contact manifold of dimension $2n - 1$.*

Proof. Since k is nowhere-vanishing, around each point one can form a local leaf space, M' , for the one-dimensional distribution spanned by k . Further, since the conformal Killing field k corresponds to a parallel adjoint tractor, it inserts trivially into the curvature of the normal conformal Cartan connection, see Corollary 3.5 in [Čap08]. By the discussion above, the conformal Cartan geometry $(\tilde{\mathcal{G}} \rightarrow \tilde{M}, \tilde{\omega}^{nor})$ reduces to a Cartan geometry $(\mathcal{G} \rightarrow \tilde{M}, \omega)$ of type (G, Q) . Hence k inserts trivially into the curvature of ω , i.e. $i_k \Omega = 0$. Since, under the identification $T\tilde{M} = \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$, the distribution spanned by k corresponds to $\mathfrak{p}'/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$, this is precisely the curvature restriction from Corollary 2.7 in [Čap05]. By connectedness of P'/Q , we can thus conclude that M' inherits a Cartan geometry of type (G, P') , and the correspondence space of that geometry is locally isomorphic to \tilde{M} . Since $\tilde{\omega}^{nor}$ is torsion-free and $\mathfrak{g} \cap \tilde{\mathfrak{p}} \subset \mathfrak{p}'$, the parabolic geometry $(\mathcal{G} \rightarrow M', \omega)$ is torsion-free and regular. Consequently, it determines a torsion-free Lagrangean contact

structure on M' . The Fefferman-type construction discussed in this section then recovers the original conformal structure. \square

Remark 4.13. By Theorem 3.1 in [ČG10], if $\mathbf{K} \in \Gamma(\widetilde{\mathcal{AM}})$ is a parallel section and the underlying conformal Killing field k is light-like, then $\mathbf{K}^2 = \lambda \text{id}$ where λ is a constant explicitly given as

$$\lambda = \frac{1}{(2n)^2} (\widetilde{D}_a k^a)^2 - k^a \widetilde{\mathbf{P}}_{ab} k^b - \frac{1}{2n} k^a \widetilde{D}_a \widetilde{D}_b k^b$$

(in our notation this is the trace $\langle \mathbf{K}, \mathbf{K} \rangle$ up to a positive multiple). This allows us to rephrase the characterisation result Proposition 4.12 in underlying terms: A split-signature conformal structure $(\widetilde{M}, \mathbf{c})$ is locally induced by a torsion-free Lagrangean contact structure if and only if it admits a light-like conformal Killing field k that inserts trivially into Weyl and Cotton tensor such that λ as defined above is positive.

4.5. Exceptional case: Dimension $n = 2$. Conformal structures induced from 2-dimensional projective structures are special and well-studied see e.g. [NS03], [DT10], [CS07]. The fact that for $n = 2$ normality is preserved (see Proposition 4.3) implies that the induced 4-dimensional conformal structures satisfy the condition from Proposition 4.10 with nowhere-vanishing underlying twistor spinors and conformal Killing field. Moreover, the fact that they come from projective structures immediately implies that the induced curvature must be horizontal, i.e. $i_\xi \widetilde{\Omega} = 0$ for all $\xi \in \Gamma(V\widetilde{M})$, where $V\widetilde{M}$ is the vertical bundle for the projection $\widetilde{M} \rightarrow M$. Equivalently, $V\widetilde{M} = \widetilde{f}$ is the kernel of the induced twistor spinor χ .

Proposition 4.14. *A conformal structure of signature $(2, 2)$ is locally the induced conformal structure on the Fefferman space over a 2-dimensional projective structure if and only if the properties from Corollary 4.11 hold and the curvature of the normal conformal Cartan connection satisfies*

$$i_\xi \widetilde{\Omega} = 0, \quad \text{for all } \xi \in \Gamma(\widetilde{f}). \quad (40)$$

Proof. The properties from Corollary 4.11 imply that the normal conformal Cartan geometry $(\widetilde{\mathcal{G}}, \widetilde{\omega})$ can be reduced to a Cartan geometry (\mathcal{G}, ω) of type (G, Q) . This Cartan geometry is locally the correspondence space over a Cartan geometry of type (G, P) , i.e. a projective geometry, if and only if $i_\xi \Omega = 0$ for all $\xi \in \Gamma(\mathcal{G} \times_Q \mathfrak{p}/\mathfrak{q})$, see [Čap05], and $\mathcal{G} \times_Q \mathfrak{p}/\mathfrak{q} = \widetilde{f}$. \square

Remark 4.15. For $n = 2$ the intermediate 3-dimensional Lagrangean contact structure can be equivalently viewed as a path geometry (or geometry of second order ODEs modulo point transformations). This geometry comes from a projective structure, i.e. the paths are the unparametrised geodesics of a projective class of connections, if and only if one of the two harmonic curvatures vanishes. It is shown in [NS03, Nur05] that this is equivalent to vanishing of a part (self dual or anti-selfdual) of the Weyl curvature of the induced conformal structure. In particular, the condition that $i_\xi \widetilde{\Omega} = 0$ for all $\xi \in \Gamma(\widetilde{f})$ can be replaced by the condition that the conformal structure be anti-selfdual.

5. NORMALISATION AND CHARACTERISATION FOR GENERAL DIMENSION

By Proposition 4.3, for $n \geq 3$, the induced conformal Cartan connection associated to a non-flat n -dimensional projective structure differs from the normal conformal Cartan connection for the induced conformal structure. In this section we will analyse the form of the difference and thus derive properties of the induced conformal structures. Furthermore, we will show that any split-signature conformal manifold having these properties is locally equivalent to the conformal structure on the Fefferman space over a projective manifold.

5.1. The normalisation process. We are going to normalise the Cartan connection $\tilde{\omega}^{ind} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ that is induced by the projective Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Any other conformal Cartan connection $\tilde{\omega}'$ differs from $\tilde{\omega}^{ind}$ by some $\Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$: $\tilde{\omega}' = \tilde{\omega}^{ind} + \Psi$. This Ψ must vanish on vertical fields and be \tilde{P} -equivariant. The condition on $\tilde{\omega}'$ to induce the same conformal structure on \tilde{M} as $\tilde{\omega}^{ind}$ is that Ψ has values in $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. One can therefore regard Ψ as a \tilde{P} -equivariant function $\Psi : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$.

The general theory of parabolic geometries, see e.g. [ČS09, Section 3.1.13], tells us that there is a unique such Ψ such that the curvature function $\tilde{\kappa}'$ of $\tilde{\omega}'$ satisfies $\tilde{\partial}^* \tilde{\kappa}' = 0$, and then $\tilde{\omega}'$ is the normal conformal Cartan connection $\tilde{\omega}^{nor}$.

The failure of $\tilde{\omega}^{ind}$ to be normal is given by $\tilde{\partial}^* \tilde{\kappa}^{ind} : \mathcal{G} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. The normalisation of $\tilde{\omega}^{ind}$ proceeds by homogeneity of $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$, which decomposes into two homogeneous components according to the decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. In the first step of normalisation one looks for a Ψ^1 such that $\tilde{\omega}^1 = \tilde{\omega} + \Psi^1$ has $\tilde{\partial}^* \tilde{\kappa}^1$ taking values in the highest homogeneity $\tilde{\partial}^* \tilde{\kappa}^1 : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$.

To write down this first normalisation we employ a Weyl structure $\tilde{\mathcal{G}}_0 \xrightarrow{j} \tilde{\mathcal{G}}$, and by Proposition 4.4 we can take a reduced Weyl structure, i.e. one that is induced by a Q_0 -reduction

$$\mathcal{G}_0 \xrightarrow{j} \mathcal{G} \hookrightarrow \tilde{\mathcal{G}}.$$

This allows us to project $\tilde{\partial}^* \tilde{\kappa}^{ind}$ to $(\tilde{\partial}^* \tilde{\kappa}^{ind})_0 : \mathcal{G}_0 \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and to employ the \tilde{G}_0 -equivariant Kostant Laplacian $\tilde{\square} : (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0 \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$, $\tilde{\square} := \tilde{\partial} \circ \tilde{\partial}^* + \tilde{\partial}^* \circ \tilde{\partial}$. For the first normalisation step we need to form a map $\Psi^1 : \tilde{\mathcal{G}} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ that agrees with $-\tilde{\square}^{-1}(\tilde{\partial}^* \tilde{\kappa}^{ind})_0$ in the $\tilde{\mathfrak{g}}_0$ -component. If we have formed any such Ψ^1 along $\mathcal{G}_0 \xrightarrow{j} \tilde{\mathcal{G}}$ we can just equivariantly extend this to all of $\tilde{\mathcal{G}}$.

To proceed with the analysis of the normalisation we need to establish a couple of technical Lemmas.

Lemma 5.1. *Let $V \subset \tilde{V}$ be a \mathfrak{g} -representation contained in a $\tilde{\mathfrak{g}}$ -representation and denote by $\phi \mapsto \tilde{\phi}$ the natural inclusion*

$$\Lambda^k \mathfrak{p}_+ \otimes V \hookrightarrow \Lambda^k \tilde{\mathfrak{p}}_+ \otimes \tilde{V}.$$

Then for any $\phi \in \Lambda^k \mathfrak{p}_+ \otimes V$

$$\widetilde{\partial^* \phi} - \widetilde{\partial^*} \widetilde{\phi} \in \Lambda^{k-1} \mathfrak{p}_+ \otimes (\Lambda^2 F \bullet V) \subset \Lambda^{k-1} \widetilde{\mathfrak{p}}_+ \otimes \widetilde{V},$$

In particular, for $V = \mathfrak{g}$, $\partial^* \phi = 0$ if and only if $\widetilde{\partial^*} \widetilde{\phi}$ takes values in $\Lambda^2 F$.

Proof. For the sake of presentation, assume that ϕ is decomposable, i.e. of the form

$$\phi = Z_0 \wedge \cdots \wedge Z_k \otimes v,$$

where $Z_0, \dots, Z_k \in \mathfrak{p}_+$ and $v \in V$. Let $\widetilde{Z}_0, \dots, \widetilde{Z}_k \in f[-2] \subset \widetilde{\mathfrak{p}}_+$ be the images of these elements under $\varphi : \mathfrak{p}_+ \rightarrow \widetilde{\mathfrak{p}}_+$. Then the differences $\widetilde{Z}_i - Z_i$ are represented by the matrices as in (68) where only the Z -entries are non-vanishing and hence contained in $\Lambda^2 F$. By the standard formula for the Kostant co-differential, see e.g. [ČS09],

$$\widetilde{\partial^* \phi} - \widetilde{\partial^*} \widetilde{\phi}$$

evaluated on any $Y_1, \dots, Y_k \in (\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{p}})$ is a linear combination of terms of the form

$$(\widetilde{Z}_i - Z_i) \bullet v,$$

which are contained in the image of $\bullet : \Lambda^2 F \times V \rightarrow V$. Thus, the first claim follows. For the second claim note that $\Lambda^2 F$ is preserved by Lie brackets with $\mathfrak{g} \subset \widetilde{\mathfrak{g}}$. \square

Lemma 5.2. *Let $\widetilde{\omega}^{ind}$ be the conformal Cartan connection induced from a normal projective Cartan connection via the Fefferman-type construction.*

(a) *Then, for any $u \in \mathcal{G}$,*

$$\widetilde{\partial^*} \widetilde{\kappa}^{ind}(u) \in f \otimes \Lambda^2 \bar{F}[-2], \quad (41)$$

and

$$(\widetilde{\partial^*} \widetilde{\kappa}^{ind})_0(u) \in f \odot \Lambda^2 f[-4], \quad (42)$$

i.e. $(\widetilde{\partial^*} \widetilde{\kappa}^{ind})_0(u)$ is contained in the kernel of the alternation map

$$\text{alt} : f \otimes \Lambda^2 f[-4] \rightarrow \Lambda^3 f[-4].$$

(b) $\Psi^1 := -\frac{1}{2} \widetilde{\partial^*} \widetilde{\kappa}^{ind}$ is a modification map accomplishing the first normalisation step.

Proof. (a) We consider the composition of the following maps

$$\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \hookrightarrow \Lambda^2 \widetilde{\mathfrak{p}}_+ \otimes \widetilde{\mathfrak{g}} \rightarrow \widetilde{\mathfrak{p}}_+ \otimes \widetilde{\mathfrak{p}} \rightarrow \widetilde{\mathfrak{p}}_+ \otimes \widetilde{\mathfrak{p}}/\widetilde{\mathfrak{p}}_+, \quad (43)$$

where the first map is $\Lambda^2 \varphi \otimes i'$ as in (27), the second map is the Kostant co-differential $\widetilde{\partial^*}$, and the last map is induced by the projection $\widetilde{\mathfrak{p}} \rightarrow \widetilde{\mathfrak{p}}/\widetilde{\mathfrak{p}}_+ \cong \widetilde{\mathfrak{g}}_0$. Applying the composition of the first two maps to $\kappa(u)$ gives $\widetilde{\partial^*} \widetilde{\kappa}^{ind}(u)$ and projecting we get $(\widetilde{\partial^*} \widetilde{\kappa}^{ind})_0(u)$. We have $\varphi(\mathfrak{p}_+) = f[-2] \subset (\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{p}})^*$, see (74), and by Lemma 5.1, $\widetilde{\partial^*} \widetilde{\kappa}^{ind}$ takes values in $\Lambda^2 F \cap \widetilde{\mathfrak{p}} = \Lambda^2 \bar{F}$. Thus, (41) holds. The image of $\Lambda^2 \bar{F}$ under the projection $\widetilde{\mathfrak{p}} \rightarrow \widetilde{\mathfrak{p}}/\widetilde{\mathfrak{p}}_+ \cong \widetilde{\mathfrak{g}}_0$ is $\Lambda^2 f[-2]$ and so $\widetilde{\partial^*} \widetilde{\kappa}_0^{ind}(u) \in f \otimes \Lambda^2 f[-4]$.

Next we show that $\widetilde{\partial^*} \widetilde{\kappa}_0^{ind}(u)$ is contained in the kernel of the alternation map alt . For that purpose, consider the simple part of Q_0 , i.e. the subgroup

$\mathrm{SL}(n-1) \subset Q$, which in the matrix realisation (73) corresponds to the A -block of the Lie algebra \mathfrak{q}_0 . It includes into $\widetilde{G}_0 = \mathrm{SO}(n, n)$, and the map (43) from above is $\mathrm{SL}(n-1)$ -equivariant with respect to that inclusion. The alternation map is equivariant as well. In particular, we know that $\kappa \mapsto \mathrm{alt}(\tilde{\partial}^* \tilde{\kappa}_0^{\mathrm{ind}})$ is either zero or an isomorphism on each $\mathrm{SL}(n-1)$ -irreducible component of $\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$.

Note that any element $\phi \in \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{p}_+$ will be mapped to zero, since $i'(\mathfrak{p}_+) \subset \tilde{\mathfrak{p}}$ and so $\sum_{i=1}^n [i' \circ \phi(X_i, X), \tilde{Z}_i] \subset [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}_+] = \tilde{\mathfrak{p}}_+$, and thus the projection onto $\tilde{\mathfrak{p}}/\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{g}}_0$ vanishes. Thus it suffices to consider the space of projective Weyl curvatures, which is contained in $\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}_0$. Decomposing this space as well as the co-domain $\Lambda^3 \mathbb{R}^{n*}$ under $\mathrm{SL}(n-1)$ we see that there is only one irreducible $\mathrm{SL}(n-1)$ component that occurs in both spaces, and it is isomorphic to $\Lambda^2 \mathbb{R}^{n-1*}$.

Hence it suffices to compute the composition of (43) with the alternation map explicitly on one element of the space of Weyl tensors contained in this $\mathrm{SL}(n-1)$ -module isomorphic to $\Lambda^2 \mathbb{R}^{n-1*}$. Let $X_n \in \mathfrak{g}_-$ and $Z^n \in \mathfrak{p}_+$ be the two dual basis vectors stabilised by $\mathrm{SL}(n+1)$. Then a typical element is of the form

$$Z^1 \wedge Z^2 \otimes X_n \otimes Z^n - Z^1 \wedge Z^2 \otimes X_1 \otimes Z^1 + Z^n \wedge Z^2 \otimes X_n \otimes Z^1. \quad (44)$$

Applying the map (43) to the element (44) gives

$$-\tilde{Z}^1 \otimes \tilde{Z}^n \wedge \tilde{Z}^2 - \tilde{Z}^n \otimes \tilde{Z}^1 \wedge \tilde{Z}^2$$

which indeed lies in the kernel of the alternation map.

(b) The Kostant Laplacian $\tilde{\square}$ restricts to an invertible endomorphism of $((\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \mathrm{im} \tilde{\partial}^*$ that acts by scalar multiplication on each of the three \widetilde{G}_0 -irreducible components. Now it was shown above that $(\tilde{\partial}^* \tilde{\kappa}^{\mathrm{ind}})_0$ is contained in one of the irreducible components, namely in

$$(\tilde{\partial}^* \tilde{\kappa}^{\mathrm{ind}})_0(u) \in f \odot \Lambda^2 f[-4]$$

and it was computed in [Ham08] that on this component $\tilde{\square}$ acts by multiplication by 2. Thus,

$$-\tilde{\square}^{-1}(\tilde{\partial}^* \tilde{\kappa}^{\mathrm{ind}})_0 = \frac{1}{2}(\tilde{\partial}^* \tilde{\kappa}^{\mathrm{ind}})_0. \quad \square$$

As a first non-trivial property of the induced conformal structures we can now show the existence of a canonical twistor spinor.

Proposition 5.3. *The spin tractor $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-)$ from Proposition 4.2 is parallel with respect to the normal conformal spin tractor connection, i.e. $\tilde{\nabla}^{\mathrm{nor}} \mathbf{s}_F = 0$. In particular, the conformal spin structure (M, \mathbf{c}) carries a canonical (pure) twistor spinor $\chi \in \Gamma(\tilde{\mathcal{S}}_-[\frac{1}{2}])$.*

Proof. From Lemma 5.2 we know the first normalisation step

$$\Psi^1 : \mathcal{G} \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \Lambda^2 \bar{F} \subset (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}.$$

We set $\tilde{\omega}^1 = \tilde{\omega} + \Psi^1$. Since $F \subset \mathbb{R}^{n+1, n+1}$ is the kernel of the pure spinor $\mathbf{s}_F \in \Delta_-^{n+1, n+1}$ we see that the tractor spinor \mathbf{s}_F induced by the constant

map

$$\mathcal{G} \rightarrow \Delta_-^{n+1, n+1}, \quad u \mapsto s_F$$

is still parallel with respect to $\tilde{\omega}^1$.

Now we want to see that also after the second normalisation step, which yields the normal conformal Cartan connection $\tilde{\omega}^2 = \tilde{\omega}^{nor}$, the tractor spinor \mathbf{s}_F is still parallel. One has $\tilde{\omega}^{nor} = \tilde{\omega}^1 + \Psi^2$, with $\Psi^2 : \mathcal{G}_0 \rightarrow \tilde{\mathfrak{p}}_+$, and we denote the spin tractor connections on $\tilde{\mathcal{S}}_-$ induced by $\tilde{\omega}^1$ and $\tilde{\omega}^{nor}$ by $\tilde{\nabla}^1$ resp. $\tilde{\nabla}^{nor}$.

Recall from 2.3.2 that

$$[\tilde{\mathcal{S}}_-]_g = \begin{pmatrix} \tilde{\mathcal{S}}_+[-\frac{1}{2}] \\ \tilde{\mathcal{S}}_-[\frac{1}{2}] \end{pmatrix}.$$

Now, since Ψ^2 takes values in $\tilde{\mathfrak{p}}_+$, we know that $\Psi^2 \mathbf{s}_F = \tilde{\nabla}^{nor} \mathbf{s}_F - \tilde{\nabla}^1 \mathbf{s}_F$ is contained in top slot $\Gamma(\tilde{\mathcal{S}}_+[-\frac{1}{2}]) \subset \Gamma(\tilde{\mathcal{S}}_-)$. Since \mathbf{s}_F is parallel with respect to $\tilde{\nabla}^1$ this implies that actually

$$\tilde{\nabla}^{nor} \mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_+[-\frac{1}{2}]). \quad (45)$$

Let $\mathbf{s}_F = \begin{pmatrix} \tilde{\chi} \\ \chi \end{pmatrix} \in \Gamma(\tilde{\mathcal{S}}_-)$. Then (45) says explicitly that

$$\begin{pmatrix} \tilde{D}_c \tilde{\chi} + \frac{1}{\sqrt{2}} \tilde{\mathbf{P}}_{cp} \gamma^p \tilde{\chi} \\ \tilde{D}_c \chi + \frac{1}{\sqrt{2}} \gamma_c \tilde{\chi} \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix}.$$

It follows in particular that χ is a twistor spinor, and necessarily $\tilde{\chi} = \frac{1}{\sqrt{2n}} \not{D} \chi$. This shows that

$$\mathbf{s}_F = L_0^{\tilde{\mathcal{S}}_-}(\chi).$$

But then $\tilde{D}_c \tilde{\chi} + \frac{1}{\sqrt{2}} \tilde{\mathbf{P}}_{cp} \gamma^p \tilde{\chi} = 0$ is a differential consequence of that equation, and thus indeed

$$0 = \tilde{\nabla}^{nor}(L_0^{\tilde{\mathcal{S}}_-} \chi) = \tilde{\nabla}^{nor} \mathbf{s}_F.$$

□

It immediately follows from the fact that \mathbf{s}_F is parallel with respect to the normal tractor connection $\tilde{\nabla}^{nor}$ that:

Corollary 5.4. *a) The curvature $\tilde{\Omega}^{nor}$ of $\tilde{\nabla}^{nor}$ annihilates \mathbf{s}_F ,*

$$\tilde{\Omega}^{nor} \in \Omega^2(\tilde{M}, (\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}})_0 \oplus \Lambda^2(\tilde{\mathcal{F}})).$$

b) The conformal holonomy of \mathbf{c} is contained in the isotropy subgroup of $s_F \in \Delta_-^{n+1, n+1}$ in $\text{Spin}(n+1, n+1)$,

$$\text{Hol}(\mathbf{c}) = \text{SL}(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subset \text{Spin}(n+1, n+1).$$

c) The pairing of $\tilde{\Omega}^{nor}$ with \mathbf{K} vanishes,

$$\langle \tilde{\Omega}^{nor}, \mathbf{K} \rangle = 0. \quad (46)$$

Recall that for any given exact reduced Weyl structure, the normal conformal tractor connection decomposes as $\tilde{\nabla} = \tilde{\partial} \oplus \tilde{D} \oplus \tilde{P}$, where $\tilde{\partial}$ is the Lie algebra differential, \tilde{D} is the Levi Civita connection and \tilde{P} is the Schouten tensor for the corresponding metric, see (7). Now, $\tilde{\nabla} = \tilde{\nabla}^{ind} + \Psi$ and we have seen in the Proposition above that Ψ has values in $(\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}})_0 \oplus \Lambda^2 \tilde{\mathcal{F}}$. Hence it follows immediately from the fact that \tilde{D}^{ind} preserves $\tilde{\mathcal{F}}$ and $\tilde{f}[-2]$ (see Remark 4.6) that also \tilde{D} preserves those bundles. Moreover, since the induced Schouten tensor \tilde{P}^{ind} has values in $\tilde{f}[-2]$ and Ψ has values in $\Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]$, it also immediately follows that also \tilde{P} has values in $\tilde{f}[-2]$. We will use these facts in the proofs of the following lemmas.

Lemma 5.5. *The difference $\Psi = \tilde{\omega}^{nor} - \tilde{\omega}^{ind}$ is horizontal and takes values in $\Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]$,*

$$\Psi \in \Omega_{hor}^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]) \subset \Omega_{hor}^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}} \oplus (\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}})_0),$$

where we use the inclusion $\tilde{f}[-2] \subset \Lambda^2 \tilde{\mathcal{F}} \oplus (\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}})_0$.

Proof. We know from the proof of Proposition 5.3 that Ψ takes values in $\Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]$. The only thing left to show is that Ψ is horizontal, i.e., $i_v \Psi = 0$ for all vertical fields v . Since the first modification Ψ^1 is a multiple of $\tilde{\partial}^* \tilde{\Omega}$, it is clear that Ψ^1 is horizontal. We employ a reduced Weyl structure and write the normal tractor connection as $\tilde{\nabla} = \tilde{\partial} \oplus \tilde{D} \oplus \tilde{P}$. We have the first normalisation modification $\Psi^1 \in \Omega_{hor}^1(\tilde{M}, \Lambda^2(\tilde{\mathcal{F}}))$. Then we have, using the notation of section 3.2, that for vertical ξ

$$\begin{aligned} \tilde{\partial}^* d^{\tilde{D}} \Psi^1(\xi) = \\ \sum (\tilde{Z}_i \bullet \tilde{D}_\xi \Psi^1(\tilde{X}_i) - \tilde{Z}_i \bullet \Psi^1(\tilde{D}_\xi \tilde{X}_i) - \tilde{Z}_i \bullet \tilde{D}_{\tilde{X}_i} \Psi^1(\xi) + \tilde{Z}_i \bullet \Psi^1(\tilde{D}_{\tilde{X}_i} \xi)), \end{aligned}$$

where the third and fourth part of the sum are vanishing due to horizontality of Ψ^1 . Since Ψ^1 is horizontal, we only need to consider pairs $(\tilde{X}_i, \tilde{Z}_i)$ with \tilde{X}_i not vertical. But for those \tilde{X}_i , the corresponding elements \tilde{Z}_i bracket trivially with $\Lambda^2 \tilde{\mathcal{F}}$, and since \tilde{D}_ξ preserves $\tilde{\mathcal{F}}$ also the first and second sum vanish. Therefore $\tilde{\partial}^* d^{\tilde{D}} \Psi^1$, and then also Ψ^2 , lie in $\Omega_{hor}^1(\tilde{M}, \tilde{f}[-2])$. \square

Lemma 5.6. *The normal conformal Cartan curvature $\tilde{\Omega}^{nor}$ satisfies the integrability condition*

$$i_v \tilde{\Omega}^{nor} \in \Omega_{hor}^1(\tilde{M}, \Lambda^2(\tilde{\mathcal{F}}) \oplus \tilde{f}[-2]). \quad (47)$$

It follows in particular that $i_v i_w \tilde{\Omega}^{nor} = 0$ for $v, w \in \Gamma(\tilde{f}[-2])$.

Proof. We have

$$\tilde{\Omega}^{nor} = \tilde{\Omega} + d^{\tilde{\nabla}} \Psi + [\Psi, \Psi].$$

From the specific form of Ψ it immediately follows that $[\Psi, \Psi] = 0$. Since $\tilde{\Omega}$ is the curvature of the induced connection, it is automatically horizontal. So let us consider the terms of the form $i_v d^{\tilde{\nabla}} \Psi$ for vertical v : Since Ψ is horizontal, we only need to consider $\tilde{\nabla}_v \Psi(\xi)$. If ξ is vertical, this expression

vanishes, and thus $i_v \tilde{\Omega}^{nor}$ is horizontal. Thus we only need to verify that $\tilde{\nabla}_v \Psi(\xi)$ lies in $\Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]$ for every ξ . For this, we again employ a reduced Weyl structure and write $\tilde{\nabla} = \tilde{\partial} \oplus \tilde{D} \oplus \tilde{P}$. But \tilde{D}_v preserves $\tilde{\mathcal{F}}$ and $\tilde{f}[-2]$ and $\tilde{P}(v)$ (since it takes values in $\tilde{f}[-2]$) and $\tilde{\partial}_v$ act trivially on $\Lambda^2 \tilde{\mathcal{F}} \oplus \tilde{f}[-2]$. \square

Next we study the question whether the tractors \mathbf{s}_E , \mathbf{s}_F and \mathbf{K} are given by BGG-splitting operators L_0 from their underlying objects. Recall, that this means to verify

$$\tilde{\partial}^* \tilde{\nabla}^{nor} \mathbf{s}_F = 0, \quad \tilde{\partial}^* \tilde{\nabla}^{nor} \mathbf{s}_E = 0 \quad \text{and} \quad \tilde{\partial}^* \tilde{\nabla}^{nor} \mathbf{K} = 0,$$

see (4). This obviously holds for the spin tractor \mathbf{s}_F since $\tilde{\nabla}^{nor} \mathbf{s}_F = 0$. Hence $\mathbf{s}_F = L_0^{\tilde{\mathcal{S}}_-}(\chi)$ where $\chi \in \Gamma(S_-[\frac{1}{2}])$ is the underlying twistor spinor.

Proposition 5.7. *Let \mathbf{s}_E and \mathbf{K} be the tractor spinor and adjoint tractor from Proposition 4.2 with underlying objects $\eta \in \Gamma(\tilde{\mathcal{S}}_{\pm}[\frac{1}{2}])$ and $k \in \Gamma(T\tilde{M})$. Then*

- (i) $\mathbf{s}_E = L_0^{\tilde{\mathcal{S}}_{\pm}}(\eta)$,
- (ii) $\mathbf{K} = L_0^{\Lambda^2 \tilde{\mathcal{T}}}(k)$.

Proof. We can decompose $\Psi : \mathcal{G}_0 \rightarrow \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \otimes \tilde{\mathfrak{g}}$ as

$$\Psi = \Psi_{\tilde{\mathfrak{g}}_0} + \Psi_{\tilde{\mathfrak{p}}_+}.$$

According to 5.2 and 5.5 we have that $\Psi_{\tilde{\mathfrak{g}}_0}$ takes values in $f \odot \Lambda^2 f[-4]$ and $\Psi_{\tilde{\mathfrak{p}}_+}$ takes values in $f \otimes f[-4]$.

(i) The tractor spinor \mathbf{s}_E is of the form $\mathbf{s}_E = \begin{pmatrix} * \\ \eta \end{pmatrix} \in \tilde{\mathcal{S}}_{\pm}$, and since $\tilde{\nabla}^{nor} \mathbf{s}_E = \Psi \bullet \mathbf{s}_E$ we have

$$\tilde{\partial}^* \tilde{\nabla}^{nor} \mathbf{s}_E = \tilde{\partial}^* \Psi \bullet \mathbf{s}_E = \begin{pmatrix} (\Psi_{\tilde{\mathfrak{g}}_0} \otimes \eta)_{\tilde{\mathcal{S}}_{\mp}[-\frac{1}{2}]} \\ 0 \end{pmatrix}.$$

Here the projection $(\Psi_{\tilde{\mathfrak{g}}_0} \otimes \eta)_{\tilde{\mathcal{S}}_{\mp}[-\frac{1}{2}]}$ can be realised as the full (triple) Clifford action on $\Psi_{\tilde{\mathfrak{g}}_0} \in \bigotimes^3 \tilde{f}[-4]$. Now it is easy to see that this action must vanish for a $\Psi_{\tilde{\mathfrak{g}}_0} \in \tilde{f} \odot \Lambda^2 \tilde{f}[-4]$: e.g., realise $\Psi_{\tilde{\mathfrak{g}}_0}$ equivalently in $S^2 \tilde{f} \otimes \tilde{f}[-4]$ by symmetrisation in the first two slots, then the complete Clifford action on η vanishes because the action of the first two slots is just a (trivial) trace multiplication.

(ii) The compatibility (33) can be rephrased as $\mathbf{K} = (\mathbf{s}_E \otimes \mathbf{s}_F)_{\Lambda^2 \tilde{\mathcal{T}}}$. Hence it is sufficient to show that $\mathbf{s}_E \otimes \mathbf{s}_F$ is given by the BGG-splitting. Since $\tilde{\nabla}^{nor} \mathbf{s}_F = 0$, $\tilde{\nabla}^{nor} \mathbf{s}_E = \Psi \bullet \mathbf{s}_E$, and Ψ is horizontal, we have

$$\tilde{\nabla}^{nor}(\mathbf{s}_E \otimes \mathbf{s}_F) = (\Psi \bullet \mathbf{s}_E) \otimes \mathbf{s}_F \in \Gamma(\tilde{f}[-2] \otimes \tilde{\mathcal{S}}_{\pm} \otimes \tilde{\mathcal{S}}_-).$$

Hence $\tilde{\partial}^* \tilde{\nabla}^{nor}(\mathbf{s}_E \otimes \mathbf{s}_F)$ is given by the action of $\tilde{f}[-2] \subset T^* \tilde{M}$ on $\Psi \bullet \mathbf{s}_E$ and \mathbf{s}_F . The former term vanishes using the previous condition (i), the latter term vanishes since $\tilde{f}[-2]$ acts trivially on \mathbf{s}_F . \square

Corollary 5.8. *The vector field $k \in \Gamma(T\widetilde{M})$ from Proposition 4.2 is a conformal Killing field. In particular,*

$$\widetilde{\nabla}^{nor}\mathbf{K} = i_k\widetilde{\Omega}^{nor}. \quad (48)$$

Moreover,

$$i_k\widetilde{\Omega}^{nor} = 2\Psi_{\Lambda^2 F}. \quad (49)$$

Proof. Since $\mathbf{K} = L_0^{\Lambda^2\widetilde{\mathcal{T}}}(k)$, k is a conformal Killing field if and only if $\widetilde{\nabla}^{nor}\mathbf{K}$ has trivial projecting slot. Now $\widetilde{\nabla}^{ind}\mathbf{K} = 0$, and thus $\widetilde{\nabla}^{nor}\mathbf{K} = -\Psi\bullet\mathbf{K}$. But the projecting slot of $\Psi\bullet\mathbf{K}$ is obtained by the action of $\Psi_{\widetilde{\mathfrak{g}}_0} \in \widetilde{f} \odot \Lambda^2\widetilde{f}[-4]$ on $k \in \widetilde{e} \cap \widetilde{f} \subset \widetilde{f}$, which vanishes. The display (48) is a well known property of conformal Killing fields, cf. (23).

We have $\widetilde{\nabla}^{nor}\mathbf{K} = i_k\widetilde{\Omega}^{nor}$ according to (48). On the other hand since Ψ has values in $\Lambda^2\widetilde{\mathcal{F}} \oplus (\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}})_0$ according to 5.5, we have $\mathbf{K}\bullet\Psi = -2\Psi_{\Lambda^2 F}$. Thus

$$0 = \widetilde{\nabla}^{ind}\mathbf{K} = (\widetilde{\nabla}^{nor}\mathbf{K} - \Psi\bullet)\mathbf{K} = i_k\widetilde{\Omega}^{nor} - 2\Psi_{\Lambda^2 F}$$

since $\mathbf{K}\bullet\Psi = -\Psi\bullet\mathbf{K}$. \square

We now summarise/collect all the information on the induced conformal structure $(\widetilde{M}, \mathbf{c})$ that we derived:

Proposition 5.9. *The split-signature conformal spin structures $(\widetilde{M}, \mathbf{c})$ induced by projective structures via the Fefferman-type construction $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ admit (χ, η, k) such that*

- (a) χ is a pure twistor spinor with (parallel) pure tractor spinor $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}^-}(\chi)$,
- (b) η is a pure spinor with pure tractor spinor $\mathbf{s}_E = L_0^{\widetilde{\mathcal{S}}^\pm}(\eta)$.
- (c) k is a conformal Killing field with tractor endomorphism $\mathbf{K} = L_0^{A\widetilde{M}}(k)$ which acts by id on $\ker \mathbf{s}_E$ and $-\mathrm{id}$ on $\ker \mathbf{s}_F$.

The spinors χ, η and the field k are nowhere-vanishing.

Moreover, the following integrability condition is satisfied:

$$i_v\widetilde{\Omega}^{nor} \in \Omega_{hor}^1(\widetilde{M}, \Lambda^2(\widetilde{\mathcal{F}}) \oplus \widetilde{f}[-2]) \quad (I)$$

for $v \in \Gamma(V\widetilde{M})$.

It will be useful to have the following integrability condition in terms of the Weyl tensor only, which is equivalent to condition (I) assuming the algebraic conditions of Proposition 5.9:

Lemma 5.10. *For a given split-signature (n, n) conformal structure $(\widetilde{M}, \mathbf{c})$ endowed with tractors $\mathbf{s}_E, \mathbf{s}_F$ and \mathbf{K} satisfying conditions a), b) and c), condition (I) is equivalent to the following condition on the Weyl tensor*

$$v^a w^c \widetilde{W}_{abcd} = 0 \quad (W)$$

for all $v, w \in \Gamma(V\widetilde{M})$.

Proof. The implication (I) \implies (W) is obvious. It remains to prove the converse implication (W) \implies (I). By (W), one has that $(i_v \tilde{\Omega}^{nor})_{\tilde{\mathfrak{g}}_0}$ is a one-form with values in $\Lambda^2(\tilde{\mathcal{F}})$. Under the algebraic assumptions the curvature function of $\tilde{\Omega}^{nor}$ has values in $(E \otimes F \oplus \Lambda^2 F) \cap \tilde{\mathfrak{p}}$. The projection of that space to $\tilde{\mathfrak{p}}_+$ is precisely $f[-2]$, hence it follows that $(i_v \tilde{\Omega}^{nor})_{\tilde{\mathfrak{p}}_+}$ is a one-form with values in $\tilde{f}[-2]$.

We next prove horizontality of $i_v \tilde{\Omega}^{nor}$. By (W) and the algebraic Bianchi-identity, \tilde{W}_{abcd} vanishes upon insertion of two vertical vectors anywhere; in particular $v^a \tilde{W}_{abcd}$ is horizontal. Now, as in the proof of Proposition 4.2, $u \in \Gamma(\tilde{f})$ corresponds to a section of $\tilde{\mathcal{F}} \subset \tilde{\mathcal{T}}$ of the form $\begin{pmatrix} * \\ u \\ 0 \end{pmatrix}$. By (77) the action of $i_v \tilde{\Omega}^{nor}$ on such element is of the form $\begin{pmatrix} v^r u^s \tilde{Y}_{rsa} \\ * \\ 0 \end{pmatrix}$ and this has to vanish by the properties of $\tilde{\Omega}^{nor}$, hence $v^r u^s \tilde{Y}_{rsa} = 0$. Altogether, $i_v \tilde{\Omega}^{nor}$ is horizontal. \square

5.2. Backward direction: Characterisation. We are now going to characterise the induced conformal structures. For this purpose it is useful to find, in the case where we already know the structure to be induced, a formula that describes the induced Cartan connection form $\tilde{\omega}^{ind}$ in terms of purely conformal data, i.e. $\tilde{\omega}^{nor}$ and $\tilde{\Omega}^{nor}$. While we cannot, at this stage, give a precise formula for $\tilde{\omega}^{ind}$ in terms of that conformal data, the following (intermediate) Cartan connection form will be enough for our purposes:

$$\tilde{\omega}' := \tilde{\omega}^{nor} - \frac{1}{2} i_k \tilde{\Omega}^{nor}. \quad (50)$$

In fact, the following Lemma, which follows immediately from Lemma 5.5 and (49), shows that the Cartan connection form $\tilde{\omega}'$ coincides with the induced Cartan connection up to terms in \mathfrak{p}_+ :

Lemma 5.11. *Let (\mathcal{G}, ω) be the normal Cartan geometry describing a projective structure, $(\tilde{\mathcal{G}}, \tilde{\omega}^{ind})$ the induced conformal Cartan geometry, and $(\tilde{\mathcal{G}}, \tilde{\omega}^{nor})$ the normal conformal Cartan geometry obtained by normalisation of $\tilde{\omega}^{ind}$. Then, along $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ and with $\tilde{\omega}'$ as in (50), we have*

$$(\tilde{\omega}')_{\mathfrak{g}_0} = (\tilde{\omega}^{nor})_{\mathfrak{g}_0} = (\tilde{\omega}^{ind})_{\mathfrak{g}_0}.$$

For the rest of this section, we will start with a given split-signature conformal (spin) structure (\tilde{M}, \mathbf{c}) satisfying all the properties of Proposition 5.9. In particular, \tilde{M} is endowed with a conformal Killing field $k \in \Gamma(T\tilde{M})$, and we can still use formula (50) to define a Cartan connection $\tilde{\omega}'$. The corresponding tractor connection will be denoted by $\tilde{\nabla}'$ and the curvature by $\tilde{\Omega}'$ or $\tilde{\kappa}'$.

As a first step in the backwards direction, the following proposition now shows that the so constructed Cartan connection $\tilde{\omega}'$ is in fact an $\mathrm{SL}(n+1)$ -connection.

Proposition 5.12. *Let (\tilde{M}, \mathbf{c}) be a split-signature conformal (spin) structure satisfying all the properties of Proposition 5.9. Then the sections \mathbf{s}_E ,*

\mathbf{s}_F and \mathbf{K} are parallel with respect to the tractor connection $\tilde{\nabla}'$, i.e.

$$\tilde{\nabla}'\mathbf{s}_E = 0, \quad \tilde{\nabla}'\mathbf{s}_F = 0, \quad \tilde{\nabla}'\mathbf{K} = 0. \quad (51)$$

In particular, $\text{Hol}(\tilde{\omega}') \subset \text{SL}(n+1)$ and $\tilde{\omega}'$ reduces to a Cartan connection of type $(\text{SL}(n+1), Q)$ on a reduction $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$. The curvature $\tilde{\Omega}'$ of the Cartan connection $\tilde{\omega}'$ satisfies

$$\tilde{\Omega}' = (\tilde{\Omega}^{nor})_{(E \otimes F)_0}.$$

Moreover, $\tilde{\Omega}'$ takes values in \mathfrak{p} and $i_v \tilde{\Omega}'$ takes values in \mathfrak{p}_+ .

Proof. That $\tilde{\nabla}'\mathbf{s}_F = 0$ follows immediately from the fact that $\tilde{\nabla}' - \tilde{\nabla}^{nor} = -\frac{1}{2}i_k \tilde{\Omega}^{nor}$ has values in $\Lambda^2 \tilde{\mathcal{F}}$. Since k is a conformal Killing field we have $\tilde{\nabla}^{nor}\mathbf{K} = i_k \tilde{\Omega}^{nor}$. By definition

$$\tilde{\nabla}'\mathbf{K} = \tilde{\nabla}^{nor}\mathbf{K} - \frac{1}{2}i_k \tilde{\Omega}^{nor} \bullet \mathbf{K},$$

which vanishes, since $i_k \tilde{\Omega}^{nor}$ has values in $\Lambda^2 \tilde{\mathcal{F}}$ and therefore $\frac{1}{2}i_k \tilde{\Omega}^{nor} \bullet \mathbf{K} = i_k \tilde{\Omega}^{nor}$. It follows that $\text{Hol}(\tilde{\omega}') \subset \text{SL}(n+1)$, and thus \mathbf{s}_E is parallel as well. In particular, $\tilde{\omega}'$ reduces to a Cartan connection of type $(\text{SL}(n+1), Q)$ on a Q -principal bundle $\mathcal{G} \subset \tilde{\mathcal{G}}$.

We further compute that

$$\begin{aligned} \tilde{\Omega}' &= \tilde{\Omega}^{nor} - \frac{1}{2}d\tilde{\nabla}^{nor} \cdot i_k \tilde{\Omega}^{nor} = \tilde{\Omega}^{nor} - \frac{1}{2}d\tilde{\nabla}^{nor} \tilde{\nabla}^{nor}\mathbf{K} \\ &= \tilde{\Omega}^{nor} - \frac{1}{2}\tilde{\Omega}^{nor} \bullet \mathbf{K} = \tilde{\Omega}^{nor} + \frac{1}{2}\mathbf{K} \bullet \tilde{\Omega}^{nor} = (\tilde{\Omega}^{nor})_{(E \otimes F)_0}, \end{aligned}$$

where we are again using $\tilde{\nabla}^{nor}\mathbf{K} = i_k \tilde{\Omega}^{nor}$ for the conformal Killing field k and that $\tilde{\Omega}^{nor}$ has values in $\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}} \oplus \Lambda^2 \tilde{\mathcal{F}}$.

The properties of $\tilde{\Omega}'$ then follow from the properties of $\tilde{\Omega}^{nor}$ and the projections (75), since $\Lambda^2 \tilde{F} \subset \tilde{\mathfrak{g}}_0$ by (76). \square

Next, before stating the main characterisation theorem of this section, we will discuss the following proposition which is proven via a slight generalisation of a proof in [Čap05].

Proposition 5.13. *Let $(\mathcal{G} \rightarrow \tilde{M}, \omega)$ be a Cartan geometry of type $(\text{SL}(n+1), Q)$ with curvature $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{q})^* \otimes \mathfrak{g}$. We assume that $\kappa(w_1, w_2) \in \mathfrak{p}$ for any $w_1, w_2 \in \mathfrak{g}/\mathfrak{q}$, $\kappa(v, w) \in \mathfrak{p}_+$ for any $v \in \mathfrak{p}/\mathfrak{q}$ and $w \in \mathfrak{g}/\mathfrak{q}$, and $\kappa(v_1, v_2) = 0$ for any $v_1, v_2 \in \mathfrak{p}/\mathfrak{q}$. Then \mathcal{G} is locally a P -bundle over $M = \mathcal{G}/P$ and ω defines a canonical projective structure on M .*

Proof. The fact that $i_{v_1} i_{v_2} \kappa = 0$ for all $v_1, v_2 \in \mathfrak{p}/\mathfrak{q}$ implies that \mathcal{G} is locally a P -bundle $\mathcal{G} \rightarrow M$ by [Čap05]. We will restrict \mathcal{G} to assume this globally. We define $M = \mathcal{G}/P$ and $\mathcal{G}_0 = \mathcal{G}/P_+$.

Let $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ be a G_0 -equivariant splitting. It follows from $\kappa(v, \cdot) \in \mathfrak{p}_+$, for all $v \in \mathfrak{p}/\mathfrak{q}$, that

$$\mathcal{L}_{\zeta_X} \omega = -\text{ad}(X) \circ \omega \quad \text{mod } \mathfrak{p}_+,$$

for all $X \in \mathfrak{p}$. Now define $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$, $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ and $\rho \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ via the decomposition $\sigma^*(\omega) = \theta \oplus \gamma \oplus \rho$. Since σ is G_0 -equivariant and the Lie derivative is compatible with pullbacks it follows that

$$\mathcal{L}_{\tilde{\zeta}_X}(\theta \oplus \gamma) = -\text{ad}(X) \circ (\theta \oplus \gamma)$$

for all $X \in \mathfrak{g}_0$. In particular θ and γ are G_0 -equivariant and define a (reductive) Cartan geometry $(\mathcal{G}_0 \rightarrow M, \theta \oplus \gamma)$ of type $(\mathbb{R}^n \rtimes \text{SL}(n), \text{SL}(n))$, i.e. an affine connection on M . Since by assumption Ω has values in \mathfrak{p} , $\theta \oplus \gamma$ is torsion-free and so is the affine connection.

Now take another splitting $\sigma' = \sigma \cdot \exp(\Upsilon)$ for some $\Upsilon : \mathcal{G} \rightarrow \mathfrak{p}_+$. Then since $\text{Ad}(\exp(\Upsilon))$ acts by the identity on $\mathfrak{g}_- = \mathfrak{g}/\mathfrak{p}$ one has $(r^{\exp(\Upsilon)})^* \omega = \omega$ modulo \mathfrak{p} , and thus θ is independent of the choice of splitting. Let $\sigma^*(\omega) = \theta \oplus \gamma \oplus \rho$. Then $\sigma^*(\omega) = \theta \oplus \gamma' \oplus \rho'$ and $\theta \oplus \gamma' = \text{Ad}(\exp(\Upsilon)) \circ (\theta \oplus \gamma)$ (projected to $\mathfrak{g}_- \oplus \mathfrak{g}_0$). But since $\exp(\Upsilon) \in P_+$, this shows that γ' is projectively equivalent to γ . We thus obtain a well-defined projective structure on M .

Since ω is P -torsionfree and P -equivariant modulo \mathfrak{p}_+ , it can be (uniquely) modified to a normal Cartan connection $\omega^{nor} \in \Omega^1(\mathcal{G} \rightarrow M, \mathfrak{g})$ with $\omega^{nor} - \omega \in \Omega^1(\mathcal{G}, \mathfrak{p}_+)$. In particular, each splitting $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ is in fact a Weyl structure of the projective structure on M . \square

Theorem 5.14. *A split-signature (n, n) conformal spin structure \mathbf{c} on a manifold \widetilde{M} is (locally) induced by an n -dimensional projective structure via the Fefferman-type construction if and only if the properties stated in Proposition 5.9 are satisfied.*

Proof. Let $(\widetilde{M}, \mathbf{c})$ be a conformal structure with the stated properties. Then, by Proposition 5.12, $\tilde{\omega}'$ restricts to a Q -equivariant Cartan connection form with values in $\mathfrak{sl}(n+1)$ on the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$, and the corresponding curvature $\tilde{\Omega}'$ takes values in \mathfrak{p} and $i_v \tilde{\Omega}'$ takes values in \mathfrak{p}_+ . It follows from Proposition 5.13 that $\tilde{\omega}'$ can be modified to a (P -equivariant) normal projective Cartan connection (\mathcal{G}, ω) by some modification in $\Omega^1(\widetilde{M}, \mathfrak{p}_+)$. In particular (\mathcal{G}, ω) defines a projective structure \mathbf{p} on the leaf space M . Now since $\mathfrak{p}_+ \subset \mathfrak{p}$, this also shows that the Fefferman-type construction (with normalisation) starting from (M, \mathbf{p}) then yields exactly $(\widetilde{\mathcal{G}}, \tilde{\omega}^{nor})$, and thus in underlying terms $(\widetilde{M}, \mathbf{c})$.

Conversely, starting with a projective structure (M, \mathbf{p}) , it follows from Proposition 5.9 that the induced conformal structure $(\widetilde{M}, \mathbf{c})$ has all the stated properties. According to Lemma 5.11, this structure factorises to the same original projective structure \mathbf{p} on M . Thus, the Fefferman-type construction (with normalisation) and the described factorisation are (locally) inverse to each other. \square

We will now rephrase the assumption of the characterisation theorem in terms of underlying objects. For doing that, first note the following:

Remark 5.15. Since k is a conformal Killing field, we can consider the Lie derivatives $\mathcal{L}_k \chi$ and $\mathcal{L}_k \eta$ which correspond to the tractor Lie derivatives $\mathcal{L}_{\mathbf{K}} \mathbf{s}_F$ and $\mathcal{L}_{\mathbf{K}} \mathbf{s}_E$. To compute the latter, we regard $\mathbf{K} = L_0^{AM}(k)$ as a vector field $\mathbf{K} \in \mathfrak{X}(\widetilde{\mathcal{G}})$. Then, according to [ČS09],

$$\mathcal{L}_{\mathbf{K}} \mathbf{s}_F = \tilde{\nabla}_k^{nor} \mathbf{s}_F + \mathbf{K} \bullet \mathbf{s}_F.$$

Since $\tilde{\nabla}^{nor} \mathbf{s}_F = 0$ and $\mathbf{K} \bullet \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$, we therefore have

$$\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi.$$

To compute $\mathcal{L}_k \mathbf{s}_E$, we need to consider $\tilde{\nabla}^{nor} \mathbf{s}_E = -\Psi(k) \bullet \mathbf{s}_E + \mathbf{K} \bullet \mathbf{s}_E$. Since Ψ is horizontal, this yields

$$\mathcal{L}_k \eta = \frac{1}{2}(n+1)\eta.$$

Theorem 5.16. *A split-signature (n, n) conformal spin structure \mathbf{c} on a manifold \widetilde{M} is (locally) induced by an n -dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

(a) $(\widetilde{M}, \mathbf{c})$ admits a light-like conformal Killing field k^a such that relations

$$\begin{aligned} k^a k_a &= 0, & \rho^a \rho_a &= 0, \\ \mu^a_b k^b &= \varphi k^a, & \mu^a_b \rho^b &= -\varphi \rho^a, \\ k^a \rho_a &= \varphi^2 - 1, & \mu_a^c \mu_{cb} &= g_{ab} + 2k_{(a} \rho_{b)} \end{aligned} \quad (52)$$

hold, with μ, φ and ρ as defined in (21).

(b) In addition, $(\widetilde{M}, \mathbf{c})$ admits a pure twistor spinor χ such that

$$\begin{pmatrix} \frac{1}{\sqrt{2n}} \mathcal{D} \chi \\ \chi \end{pmatrix} \in \Delta_-^{n+1, n+1}$$

is pure and the twistor spinor satisfies

$$\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi,$$

where $\mathcal{L}_k \chi$ is the Lie derivative on spinors.

(c) The integrability condition $v^r w^s \widetilde{W}_{arbs} = 0$ for all $v^r, w^s \in \Gamma(\widetilde{f})$ holds.

Proof. In the case where (\widetilde{M}, c) is induced by a projective structure it follows immediately from Proposition 5.9 that we have the conformal Killing field k , the pure twistor spinor χ and the pure spinor η with corresponding tractors \mathbf{K}, \mathbf{s}_F and \mathbf{s}_E such that $\widetilde{\mathcal{T}} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}} = \ker \mathbf{s}_E \oplus \ker \mathbf{s}_F$. The fact that \mathbf{K} is an involution corresponds precisely to equations (52) in terms of μ, φ and ρ . Moreover, \mathbf{K} acts by the identity on $\widetilde{\mathcal{E}}$ and by minus identity on $\widetilde{\mathcal{F}}$, which implies that \mathbf{K} acts by $\pm \frac{1}{2}(n+1)$ on \mathbf{s}_E resp. \mathbf{s}_F . Since $\tilde{\nabla}^{nor} \mathbf{s}_F = 0$ and $\mathcal{L}_k \mathbf{s}_F = \tilde{\nabla}_k^{nor} \mathbf{s}_F + \mathbf{K} \bullet \mathbf{s}_F$, we have that $\mathcal{L}_k \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$, and thus $\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi$. Finally, the integrability condition on the Weyl tensor holds due to Proposition 5.9.

Conversely, let us consider a conformal split-signature structure $(\widetilde{M}, \mathbf{c})$ which satisfies all the stated conditions. Let $\mathbf{s}_F = L_0^{S-}(\chi)$ and $\mathbf{K} = L_0^{\Lambda^2 \widetilde{\mathcal{T}}}(k)$. Equations (52) exactly state that \mathbf{K} is an involution, and since \mathbf{K} is skew with respect to the given split-signature tractor metric on $\widetilde{\mathcal{T}}$ this is in fact equivalent to \mathbf{K} decomposing $\widetilde{\mathcal{T}}$ into two maximally isotropic eigenspaces $\widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$ with eigenvalues ± 1 . By assumption χ is a pure twistor spinor with corresponding pure spin tractor \mathbf{s}_F . Since k is a conformal Killing field, $\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi$ is equivalent to $\mathbf{K} \bullet \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$, which immediately

implies that $\ker \mathbf{s}_F = \widetilde{\mathcal{F}}$. Since we also have the integrability condition (I) satisfied, all conditions of Theorem 5.14 are satisfied, and $(\widetilde{M}, \mathbf{c})$ is locally the Fefferman-type space of a projective structure (M, \mathbf{p}) . \square

Remark 5.17. a) Since χ is by assumption a twistor spinor, the tractor spinor $\mathbf{s}_F = \begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix} \in \Delta^{n+1, n+1}$ is parallel with respect to $\widetilde{\nabla}^{nor}$, and in particular purity of $\begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix}$ can be checked at one point. If $\bar{\chi} = 0$, this tractor spinor will be pure whenever χ is pure. Otherwise, in case $\bar{\chi} \neq 0$, purity of $\begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix}$ is equivalent to χ and $\bar{\chi}$ being pure and their kernels having $(n-1)$ -dimensional intersection, cf. [Che54] (Proposition III-1.12), [HM88], [Tag15].
b) To compute the Lie derivative of χ with respect to the conformal Killing field one may use the formula

$$\mathcal{L}_k \chi = \tilde{D}_k \chi - \frac{1}{4}(\tilde{D}_{[a} k_{b]}) \gamma^a \gamma^b \chi - \frac{1}{4n}(\tilde{D}_p k^p) \chi. \quad (53)$$

This is just the Lie derivative of the (weighted) spinor χ with respect to ξ , cf. e.g. [Lic63, Kos72, FFFG96, Ham12].

6. REDUCED SCALES

Although we obtained the desired characterisation in Theorem 5.16, we still do not know the relation between $\tilde{\omega}^{nor}$ and $\tilde{\omega}^{ind}$ in detail. One of aims of the present section is to solve this problem. In this more detailed view, reduced scales play an important role. We start with their characterisation and some computational consequences.

6.1. Characterisation of reduced scales. The notion of reduced Weyl structures and reduced scales is introduced in subsection 4.3. Here we shall find an intrinsic characterisation (i.e. using the conformal structure only) of reduced scales and discuss their properties.

As the scale bundle on the projective manifold M we may consider the positive elements in the density bundle $\mathbb{E}[1]$, which is the projecting part of the dual standard tractor bundle \mathcal{T}^* . Similarly, on the Fefferman space \widetilde{M} we take the positive elements in the density bundle $\widetilde{\mathbb{E}}[1]$, the projecting part of the conformal standard tractor bundle $\widetilde{\mathcal{T}}$. Hence for a projective scale $\rho \in \Gamma(\mathbb{E}_+[1])$ we have got the tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$; similarly, for a conformal scale $\sigma \in \Gamma(\widetilde{\mathbb{E}}_+[1])$ we have the tractor $L_0^{\widetilde{\mathcal{T}}}(\sigma) \in \Gamma(\widetilde{\mathcal{T}})$. These will be termed *scale tractors*.

On the one hand, sections of $\mathbb{E}[1] \rightarrow M$ form a subset of all sections of $\widetilde{\mathbb{E}}[1] \rightarrow \widetilde{M}$, see Lemma 4.7. On the other hand, sections of $\mathcal{T}^* \rightarrow M$ are understood as specific sections of the bundle $\widetilde{\mathcal{F}} \rightarrow \widetilde{M}$, which is a subbundle in $\widetilde{\mathcal{T}} \rightarrow \widetilde{M}$, see the generalities in subsection 3.3 and the setup of our construction in subsection 4.1. It follows that these two natural inclusions commute with the BGG-splitting operators.

Lemma 6.1. *Full arrows in the following diagram commute:*

$$\begin{array}{ccc} \Gamma(\mathcal{T}^*) & \hookrightarrow & \Gamma(\tilde{\mathcal{T}}) \\ L_0^{\mathcal{T}^*} \left(\begin{array}{c} \vdots \\ \Pi_0 \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \vdots \\ \tilde{\Pi}_0 \\ \downarrow \end{array} \right) L_0^{\tilde{\mathcal{T}}} \\ \Gamma(\mathbb{E}[1]) & \hookrightarrow & \Gamma(\tilde{\mathbb{E}}[1]) \end{array}$$

Proof. Consider a projective density $\rho \in \Gamma(\mathbb{E}[1])$ on M , the corresponding tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$, and its extension to $\tilde{\mathcal{F}} \subset \tilde{\mathcal{T}}$, which is denoted by s' . The extension of $\rho \in \Gamma(\mathbb{E}[1])$ to $\tilde{\mathbb{E}}[1]$ obviously coincides with the projection $\tilde{\Pi}_0(s')$, and it is denoted by σ . We need to show that $s' = L_0^{\tilde{\mathcal{T}}}(\sigma)$, i.e. that $\tilde{\partial}^* \tilde{\nabla}^{nor} s' = 0$.

From the construction we have

$$\tilde{\nabla}^{nor} s' = \tilde{\nabla}^{ind} s' - \Psi \bullet s', \quad (54)$$

where $\tilde{\nabla}^{ind} s' \in \Omega_{hor}^1(\tilde{M}, \tilde{\mathcal{F}})$ and Ψ is the modification term with the properties as in Lemma 5.5. Decomposing $\Psi = \Psi^1 + \Psi^2$ according to the normalisation steps, we further know that $\Psi^1 \in \Omega_{hor}^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}})$ and $\Psi^2 \in \Omega_{hor}^1(\tilde{M}, \tilde{f}[-2])$. The former property and the fact that $\tilde{\mathcal{F}}$ is isotropic yield $\Psi^1 \bullet s' = 0$, the latter property and the fact that $\tilde{f}[-2] \subset T^* \tilde{M}$ mean $\tilde{\partial}^* \Psi^2 \bullet s' = 0$. Thus we have shown that $\tilde{\partial}^* \Psi \bullet s' = 0$, hence

$$\tilde{\partial}^* \tilde{\nabla}^{nor} s' = \tilde{\partial}^* \tilde{\nabla}^{ind} s'.$$

Since $\tilde{\nabla}^{ind} s' \in \Omega_{hor}^1(\tilde{M}, \tilde{\mathcal{F}})$ is the extension of $\nabla^{nor} L_0^{\mathcal{T}^*}(\rho) \in \Omega^1(M, \mathcal{T}^*)$ and $\partial^* \nabla^{nor} L_0^{\mathcal{T}^*}(\rho) = 0$, the previous display vanishes by Lemma 5.1. Indeed, for ϕ being the frame form of $\nabla^{nor} L_0^{\mathcal{T}^*}(\rho)$, we see that $\partial^* \phi$ takes values in $\Lambda^2 F \bullet F = 0$. \square

According to these facts, reduced scales can be easily characterised in terms of the corresponding scale tractors. This is the main issue of the following proposition. Recall that $V\tilde{M} = \mathcal{G} \times_Q f \subseteq T\tilde{M}$ is the vertical subbundle of the projection $\tilde{M} \rightarrow M$ and χ is the canonical twistor spinor from Theorem 5.3.

Proposition 6.2. *Suppose (\tilde{M}, \mathbf{c}) is a conformal spin structure of signature (n, n) associated to an n -dimensional oriented projective structure (M, \mathbf{p}) via the Fefferman-type construction. Let $\sigma \in \Gamma(\tilde{\mathbb{E}}_+[1])$ be a conformal scale and let $s := L_0^{\tilde{\mathcal{T}}}(\sigma) \in \Gamma(\tilde{\mathcal{T}})$ be the corresponding scale tractor. The following statements are equivalent:*

- (a) *The scale σ is reduced.*
- (b) *The tractor s is a section of $\tilde{\mathcal{F}} \subseteq \tilde{\mathcal{T}}$ and both $\tilde{\nabla}^{nor} s$ and $\tilde{\nabla}^{ind} s$ is strictly horizontal, i.e. $v^a \tilde{\nabla}_a^{nor} s = v^a \tilde{\nabla}_a^{ind} s = 0$ for every vertical vector field $v \in \Gamma(V\tilde{M})$.*
- (c) *The tractor s is a section of $\tilde{\mathcal{F}} \subseteq \tilde{\mathcal{T}}$.*
- (d) *The twistor spinor χ is parallel with respect to the Levi-Civita connection \tilde{D} of the metric corresponding to the scale σ .*

Furthermore, in reduced scales, the Schouten tensor is strictly horizontal (i.e. it satisfies $v^a \tilde{P}_{ab} = 0$ for all $v \in \Gamma(V\tilde{M})$), hence the scalar curvature \tilde{J} vanishes.

Proof. The equivalence (a) \iff (b) follows immediately from Lemma 6.1 and Proposition 3.3.

The implication (b) \implies (c) is obvious.

The implication (c) \implies (d) follows from computation in slots which we shall do in the Levi-Civita connection \tilde{D} corresponding to the scale σ , i.e. $\tilde{D}\sigma = 0$. Since χ is a twistor spinor, we only need to show that $\tilde{D}\chi = 0$ in this scale. The condition (c) means that $s \cdot \mathbf{s}_F = 0$. Thus, according to (16), (18) using also (17), we have

$$s \cdot \mathbf{s}_F = L_0^{\tilde{T}}(\sigma) \cdot L_0^{\tilde{S}}(\chi) = \begin{pmatrix} -\frac{1}{2n}\tilde{J} \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{n\sqrt{2}}\tilde{D}\chi \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2n}\tilde{J}\chi \\ -\frac{1}{n}\tilde{D}\chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (55)$$

Finally, we prove the implication (d) \implies (b): Since $\tilde{D}\chi = 0$, the form of \mathbf{s}_F simplifies to $\mathbf{s}_F = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$. Looking at the top slot of $\tilde{\nabla}_a \mathbf{s}_F = 0$, we obtain $\tilde{P}_{ac} \gamma^c \chi = 0$ hence $\tilde{P}_{ac} \in \Gamma(S^2 V\tilde{M})$ which in particular means $\tilde{J} = 0$. Summarising, we have

$$s = L_0^{\tilde{T}}(\sigma) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{s}_F = L_0^{\tilde{S}}(\chi) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad (56)$$

respectively. Hence $s \cdot \mathbf{s}_F = 0$, i.e. s is a section of $\tilde{\mathcal{F}} \subseteq \tilde{\mathcal{T}}$.

According to (15) and the previous reasoning, we further have

$$\tilde{\nabla}_a^{nor} s = \begin{pmatrix} 0 \\ \tilde{P}_{ab} \\ 0 \end{pmatrix}$$

and hence $v^a \tilde{\nabla}_a^{nor} s = 0$ for any $v \in \Gamma(V\tilde{M})$. Due to (54) and the horizontality of Ψ , it finally follows that $v^a \tilde{\nabla}_a^{ind} s = 0$. \square

Remark 6.3. Note that in reduced scales one can derive additional properties of various curvature quantities. This is treated, in a more general vein, in subsection C.2 where we refer for details.

6.2. A refinement of the normalisation procedure. So far we discussed three Cartan connections on the Fefferman space \tilde{M} : $\tilde{\omega}^{ind}$ — the one induced by the projective normal Cartan connection ω on M , see subsection 4.2; $\tilde{\omega}^{nor}$ — the corresponding conformal normal Cartan connection, see subsection 5.1; and $\tilde{\omega}'$ — the modified auxiliary Cartan connection, see subsection 5.2. Various properties of these and derived objects are enumerated in Propositions 5.9 and 5.12; the following proposition refines the integrability conditions included there:

Proposition 6.4. *The curvatures $\tilde{\Omega}^{nor}$ and $\tilde{\Omega}'$ satisfy*

$$i_v \tilde{\Omega}^{nor} \in \Omega_{hor}^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}}), \quad i_v \tilde{\Omega}' = 0 \quad (57)$$

for $v \in \Gamma(V\tilde{M})$.

Proof. From Proposition 5.6 we know that $i_v \tilde{\Omega}^{nor} \in \Omega_{hor}^1(\tilde{M}, \Lambda^2 \tilde{F} \oplus \tilde{f}[-2])$. Note the top slot of sections of $\Lambda^2 \tilde{F}$ vanishes in reduced scales, cf. (76). Thus the part in $\tilde{f}[-2]$ corresponds to $v^r \tilde{Y}_{abr}$, which however has to vanish by (104). Hence the first statement in (57) follows. The second one follows from $\tilde{\Omega}' = (\tilde{\Omega}^{nor})_{(E \otimes F)_0}$, cf. Proposition 5.12. \square

Since $\tilde{\omega}'$ is an $\text{SL}(n+1)$ -connection on $\tilde{\mathcal{G}} \rightarrow \tilde{M}$, it is just extension of a Cartan connection, say ω' , on $\mathcal{G} \rightarrow \tilde{M}$. Now, due to (57), vertical vectors insert trivially into its curvature. But this is the standard condition on the connection ω' to be a Cartan connection also on the bundle $\mathcal{G} \rightarrow M$, i.e. to be a projective Cartan connection, cf. [Čap05].

Furthermore, we can show that the descended Cartan connection is normal, i.e. $\omega' = \omega$. To do this, we first compute $\tilde{\partial}^* \tilde{\Omega}'$ and then use the relation between the co-differentials ∂^* on M and $\tilde{\partial}^*$ on \tilde{M} discussed in Lemma 5.1.

Proposition 6.5. *The curvature $\tilde{\Omega}'$ satisfies*

$$\tilde{\partial}^* \tilde{\Omega}' = i_k \tilde{\Omega}^{nor} \in \Omega^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}}), \quad (58)$$

where k is the conformal Killing field.

Proof. We shall compute $\tilde{\partial}^* \tilde{\Omega}'$ directly. First observe that using Proposition 5.12 we have $\tilde{\Omega}' = \text{Proj}_{(E \otimes F)_0} \tilde{\Omega}^{nor} = \tilde{\Omega}^{nor} + \frac{1}{2} \mathbf{K} \bullet \tilde{\Omega}^{nor}$, hence $\tilde{\partial}^* \tilde{\Omega}' = \frac{1}{2} \tilde{\partial}^* (\mathbf{K} \bullet \tilde{\Omega}^{nor})$, because $\tilde{\partial}^* \tilde{\Omega}^{nor} = 0$. The tractor \mathbf{K} has the form

$$\mathbf{K} = \begin{pmatrix} \rho_c \\ \mu_{c_0 c_1} | \varphi \\ k_c \end{pmatrix} \quad \text{where } \mu_{ab} = \tilde{D}_{[a} k_{b]} \text{ and } \varphi = -\frac{1}{2n} \tilde{D}^r k_r \quad (59)$$

according to (21) which also provides an explicit form of ρ_a .

Using (78) we compute

$$\begin{aligned} \mathbf{K} \bullet \tilde{\Omega}_{ab}^{nor} &= \begin{pmatrix} \rho_c \\ \mu_{c_0 c_1} | \varphi \\ k_c \end{pmatrix} \bullet \begin{pmatrix} -\tilde{Y}_{dab} \\ \tilde{W}_{abd_0 d_1} | 0 \\ 0 \end{pmatrix} = \\ &= \begin{pmatrix} \rho^r \tilde{W}_{abrc} - \mu_c^r \tilde{Y}_{rab} + \varphi \tilde{Y}_{cab} \\ -2\tilde{W}_{ab}^r{}_{[c_0} \mu_{c_1]r} + 2k_{[c_0} \tilde{Y}_{c_1]ab} | k^r \tilde{Y}_{rab} \\ k^r \tilde{W}_{abrc} \end{pmatrix}. \end{aligned}$$

Further, in any reduced scale, using (79) and (95), we compute

$$\tilde{\partial}^* (\mathbf{K} \bullet \tilde{\Omega}_{ab}^{nor}) = \begin{pmatrix} 0 \\ 2k^r \tilde{W}_{rac_0 c_1} | 0 \\ 0 \end{pmatrix} = 2k^r \tilde{\Omega}_{ra}^{nor}.$$

But this result lives in $\Omega^1(\tilde{M}, \Lambda^2 \tilde{\mathcal{F}})$ due to Corollary 5.8. \square

Thus using Lemma 5.1, we have proved the following:

Theorem 6.6. *Let (\mathcal{G}, ω) be a projective normal Cartan geometry over M and let $(\tilde{\mathcal{G}}, \tilde{\omega}^{ind})$ be the conformal Cartan geometry over \tilde{M} induced via the*

Fefferman-type construction. Let $\tilde{\omega}^{nor}$ be the corresponding conformal normal Cartan connection and let $\tilde{\omega}'$ be the auxiliary Cartan connection on the same bundle defined by (50). Then $\tilde{\omega}' = \tilde{\omega}^{ind}$.

According to the definition of $\tilde{\omega}'$ in (50), we have explicit relations

$$\tilde{\omega}^{ind} = \tilde{\omega}^{nor} - \frac{1}{2}i_k \tilde{\kappa}^{nor} \quad \text{and} \quad \tilde{\nabla}^{ind} = \tilde{\nabla}^{nor} - \frac{1}{2}i_k \tilde{\Omega}^{nor} \bullet. \quad (60)$$

Moreover, the normalisation process completed in Proposition 5.3 provides Ψ such that $\tilde{\omega}^{nor} = \tilde{\omega}^{ind} + \Psi$ in two normalisation steps, $\Psi = \Psi^1 + \Psi^2$. Here the first normalisation step was $\tilde{\partial}^* \tilde{\kappa}^{ind}$ up to a scalar multiple. However since $\tilde{\omega}' = \tilde{\omega}^{ind}$, it follows from Proposition 6.5 and (50) that $\tilde{\partial}^* \tilde{\kappa}' = \tilde{\partial}^* \tilde{\kappa}^{ind}$ is, up to a constant multiple, the difference between $\tilde{\omega}^{nor}$ and $\tilde{\omega}^{ind}$. This means that already the first normalisation step completes the normalisation, i.e. $\Psi^2 = 0$.

We can also explicitly describe the curvature $\tilde{\Omega}^{ind} = \tilde{\Omega}' = (\tilde{\Omega}^{nor})_{(E \otimes F)_0}$, cf. Proposition 5.12. Choosing a reduced scale, $\tilde{D}\chi = 0$, and in particular $\tilde{\chi} = 0$, and thus the equivalent properties of Lemma C.1 hold. Thus, the slots of \mathbf{K} satisfy $\rho_a = 0$, $\varphi = -1$ and $\mu_a{}^r v_r = -v_r$ for any vertical vector $v \in \Gamma(V\tilde{M})$ using the notation as in (59).

Using also (104) and the form of $\mathbf{K} \bullet \tilde{\Omega}^{nor}$ obtained in the proof of Proposition 6.5, a short computation reveals that

$$\tilde{\Omega}_{ab}^{ind} = \tilde{\Omega}_{ab}^{nor} + \frac{1}{2}\mathbf{K} \bullet \tilde{\Omega}_{ab}^{nor} = \begin{pmatrix} -\tilde{Y}_{cab} \\ \tilde{W}_{abc_0c_1} - \tilde{W}_{ab}{}^r{}_{[c_0\mu_{c_1}]r} + k_{[c_0}\tilde{Y}_{c_1]ab} \mid 0 \\ \frac{1}{2}k^r \tilde{W}_{abrc} \end{pmatrix}. \quad (61)$$

This in particular means that $\frac{1}{2}i_k \tilde{W}$ is the torsion of the Cartan connection $\tilde{\omega}^{ind}$.

Remark 6.7. The refinement of the normalisation procedure we have discussed in this section in particular shows that in the context of section 5.2 we have in fact already $i_v \tilde{\Omega}' = 0$. In particular, for a conformal structure \mathbf{c} which is already known to be induced from a projective structure \mathbf{p} , Theorem 2.7 of [Čap05] could be employed directly for descending from \mathbf{c} to \mathbf{p} .

6.3. A refinement of the characterisation. We have characterised split-signature (n, n) conformal structures \mathbf{c} on \tilde{M} induced by an n -dimensional projective structure via the Fefferman-type construction in Theorem 5.16. Our aim here is to show that this can be equivalently formulated by the following statement:

Theorem 6.8. *A split-signature (n, n) conformal structure \mathbf{c} on a manifold \tilde{M} is induced by an n -dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

- (a) (\tilde{M}, \mathbf{c}) admits a pure twistor spinor χ with (maximally isotropic, n -dimensional) integrable kernel f .
- (b) (\tilde{M}, \mathbf{c}) admits a light-like conformal Killing field k with $k \in \Gamma(\tilde{f})$.

(c) The Lie-derivative of χ with respect to the conformal Killing field k is

$$\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi.$$

(d) The integrability condition $v^r w^s \widetilde{W}_{arbs} = 0$ for all $v^r, w^s \in \Gamma(\widetilde{f})$ holds.

To show Theorem 6.8 we employ the following result from current work in progress [HSSTZ]:

Proposition. *Let χ be a pure real twistor-spinor on a conformal pseudo-Riemannian manifold $(\widetilde{M}, \mathbf{c})$ of signature (n, n) , with associated totally isotropic n -plane distribution \widetilde{f} . Suppose \widetilde{f} is integrable. Then there exists a conformal rescaling such that locally, χ is parallel, i.e. $\widetilde{D}\chi = 0$.*

In particular, the assumption on the pure twistor spinor χ of Theorem 6.8 guarantee the existence of a suitable compatible metric as in the above proposition, and we shall assume $\widetilde{D}\chi = 0$ in the following. We now proceed to prove a first major consequence:

Lemma 6.9. *Under the assumptions of Theorem 6.8 the tractor endomorphism $\mathbf{K} = L_0^{\Lambda^2 \widetilde{\mathcal{T}}}(k) \subseteq \text{End}(\widetilde{\mathcal{T}})$ satisfies*

$$\mathbf{K}^2 = \lambda \text{id}_{\widetilde{\mathcal{T}}}.$$

for $\lambda \in C^\infty(\widetilde{M})$.

Proof. Working in a scale such that $\widetilde{D}\chi = 0$, we can use additional curvature properties. In particular, it is computed in Appendix C that $v^r \widetilde{Y}_{arb} = 0$ for any $v^a \in \Gamma(\widetilde{f})$, cf. (104). Further, using the assumption (d) of Theorem 6.8, we have $\widetilde{D}^b v^r \widetilde{W}_{arbs} k^s = 0$. Since the annihilator \widetilde{f} of χ is preserved by \widetilde{D} , this means

$$v^r \widetilde{W}_{rast} \mu^{st} = 0 \quad \text{for } v^a \in \Gamma(\widetilde{f}) \text{ and } \mu_{ab} = \widetilde{D}_{[a} k_{b]}. \quad (62)$$

We shall use the notation $\mathbf{K}^2 = \mathbf{K} \odot \mathbf{K}$ where \odot is the projection $\odot : \Lambda^2 \widetilde{\mathcal{T}} \otimes \Lambda^2 \widetilde{\mathcal{T}} \rightarrow S^2 \widetilde{\mathcal{T}}$. The main step of the proof is to show that $\mathbf{K} \odot \mathbf{K}$ is the BGG-splitting, i.e. the condition $\widetilde{\partial}^* \widetilde{\nabla}^{nor}(\mathbf{K} \odot \mathbf{K}) = 0$ is satisfied. We shall show this by the direct computation. Using (48) and (59), we have

$$\widetilde{\nabla}_a^{nor}(\mathbf{K} \odot \mathbf{K}) = 2k^r \widetilde{\Omega}_{ra}^{nor} \odot \mathbf{K} = 2 \begin{pmatrix} 0 \\ k^r \widetilde{W}_{rab_0 b_1} \mid 0 \\ 0 \end{pmatrix} \odot \begin{pmatrix} \rho_c \\ \mu_{c_0 c_1} \mid \varphi \\ k_c \end{pmatrix}.$$

Computing slots of the projection \odot using (80), we obtain

$$\widetilde{\nabla}_a^{nor}(\mathbf{K} \odot \mathbf{K}) = 2 \begin{pmatrix} 0 \\ k^r \widetilde{W}_{rasb} \rho^s \\ -k^r \widetilde{W}_{ra(b_0}{}^s \mu_{b_1)s} \mid 0 \\ 0 \end{pmatrix} \in \Gamma(\widetilde{\mathbb{E}}_a \otimes S^2 \widetilde{\mathcal{T}}),$$

where we have used $k^r \widetilde{W}_{rabs} k^s = k^r \widetilde{Y}_{arb} = 0$. For all non-zero slots, application of $\widetilde{\partial}^*$ requires taking a trace. The trace of the middle slot vanishes by (62) hence $\widetilde{\partial}^* \widetilde{\nabla}^{nor}(\mathbf{K} \odot \mathbf{K}) = 0$.

We can decompose $\mathbf{K}^2 \in \Gamma(S^2\tilde{\mathcal{T}})$ to the trace free part $(\mathbf{K}^2)_0 \in S_0^2\tilde{\mathcal{T}}$ and a (functional) multiple of the identity:

$$\mathbf{K}^2 = (\mathbf{K}^2)_0 + \lambda \text{id}_{\tilde{\mathcal{T}}}.$$

Since \mathbf{K}^2 is the BGG-splitting, both summands on the right hand side are BGG-splittings. Since the projecting slot of $(\mathbf{K}^2)_0$ is $-\frac{1}{4}k^r k_r$, cf. (80), and k^a is light-like, we have shown

$$(\mathbf{K}^2)_0 = L_0^{S^2\tilde{\mathcal{T}}}(-k^r k_r) = 0. \quad \square$$

Proof of Theorem 6.8. In the case where (\tilde{M}, \mathbf{c}) is induced by a projective structure, we refer to the first part of the proof of Theorem 5.16.

In the converse direction, we employ the Levi-Civita connection \tilde{D} from Lemma 6.9. Then in particular $\tilde{D}\chi = 0$ hence $\mathbf{s}_F = L_0^{\tilde{\mathcal{S}}^-}(\chi) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$. Thus it follows easily from (17) that the tractor spinor \mathbf{s}_F is pure since χ is pure. Then we can continue as in the second part of the proof of Theorem 5.16. In particular, we have $\mathbf{K} = L_0^{\Lambda^2\tilde{\mathcal{T}}}(k)$ which satisfies $\mathbf{K} \bullet \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$. Thus the endomorphism \mathbf{K} preserves the (totally isotropic, $(n+1)$ -dimensional) annihilator $\tilde{\mathcal{F}}$ of \mathbf{s}_F and moreover the trace of this restriction is $\text{tr}(\mathbf{K}|_{\tilde{\mathcal{F}}}) = -(n+1)$. The next ingredient is Lemma 6.9 which in particular says that $\mathbf{K}^2 = \lambda \text{id}_{\tilde{\mathcal{T}}}$ for a function $\lambda \in C^\infty(\tilde{M})$. Now fix a point $x \in \tilde{M}$ and consider the value $\lambda(x) \in \mathbb{R}$. If $\lambda(x) < 0$ then endomorphism $\mathbf{K}|_{\tilde{\mathcal{F}}}$ gives rise to a complex structure, i.e. $\text{tr}(\mathbf{K}|_{\tilde{\mathcal{F}}}) = 0$ which is a contradiction. A similar reasoning excludes the case $\lambda(x) = 0$. Thus $\lambda(x) > 1$ and $\tilde{\mathcal{T}}$ decomposes to $\tilde{\mathcal{T}}_+ \oplus \tilde{\mathcal{T}}_-$ at the point $x \in \tilde{M}$ according to eigenvalues $\pm\sqrt{\lambda(x)}$. Since \mathbf{K} is skew-endomorphism and we are in the split-signature, $\tilde{\mathcal{T}}_+$ and $\tilde{\mathcal{T}}_-$ are both totally isotropic and both of the dimension $n+1$. Using once more the condition $\text{tr}(\mathbf{K}|_{\tilde{\mathcal{F}}}) = -(n+1)$, we conclude $\tilde{\mathcal{F}} = \tilde{\mathcal{T}}_-$ at x and $\lambda(x) = 1$. Since this is true for any point $x \in \tilde{M}$, we have $\mathbf{K}^2 = \text{id}_{\tilde{\mathcal{T}}}$ and one can continue as in the proof of 5.16. \square

We remark that it follows in particular in the situation of Lemma 6.9 that in fact λ is constant and \mathbf{K}^2 is parallel.

APPENDIX A. COMPARING WITH THE WALKER-TYPE CONSTRUCTION

In the current parallel article [HSSTZ] we consider a modification of the so called Riemann extensions of affine connected spaces, first described in [PW52], so that it yields a natural conformal structure of split-signature on a weighted cotangent bundle of an oriented projective manifold. In this section we briefly introduce that construction and directly compare with the one discussed in this paper.

Let M be a smooth manifold and $p : T^*M \rightarrow M$ its cotangent bundle. The vertical subbundle $V \subset T(T^*M)$ of this projection is canonically isomorphic to T^*M . An affine connection D on M determines a complementary horizontal distribution $H \subset T(T^*M)$ that is isomorphic to TM via the tangent map of p .

Definition A.1. *The Riemann extension or the Patterson–Walker metric associated to a torsion-free affine connection D on M is the pseudo-Riemannian metric g on T^*M fully determined by the following conditions:*

- (a) *both V and H are isotropic with respect to g ,*
- (b) *the value of g with one entry from V and another entry from H is given by the natural pairing between $V \cong T^*M$ and $H \cong TM$.*

It follows that V is parallel with respect to the Levi-Civita connection of the just constructed metric. Hence Patterson–Walker metrics are special cases of Walker metrics, i.e. metrics admitting a parallel isotropic distribution.

The previous definition can be adapted to weighted cotangent bundles $T^*M(w) = T^*M \otimes \mathbb{E}(w)$, provided that M is oriented and D is special, i.e. preserving a volume form on M , which allows a trivialisation of $\mathbb{E}(w)$. It turns out that Patterson–Walker metrics induced by projectively equivalent connections are conformally equivalent if and only if $w = 2$ (interpreted as a projective weight according to the conventions from subsection 2.2). Altogether, we have got a natural split-signature conformal structure on $T^*M(2)$ induced by an oriented projective structure (M, \mathbf{p}) .

From section 4.2 we know, that $\widetilde{M} = T^*M(2) \setminus \{0\}$ is the Fefferman space of the construction occupying the main part of this article. Special affine connections from \mathbf{p} are just the exact Weyl connections of the corresponding parabolic geometry. The corresponding objects on \widetilde{M} are the reduced Weyl connections, respectively reduced scales, which correspond to distinguished metrics in the conformal class, see subsection 4.3. We are going to show that these metrics are just the Patterson–Walker metrics.

By definition, the Fefferman space is $\widetilde{M} = \mathcal{G}/Q$, where $(\mathcal{G} \rightarrow M, \omega)$ is the Cartan geometry of type (G, P) associated to the projective structure on M and $Q \subset P$ is as in Appendix B.1. Under the identification $T\widetilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$, conformally invariant objects on \widetilde{M} corresponds to Q -invariant data on $\mathfrak{g}/\mathfrak{q}$. Objects related to the choice of a reduced Weyl structure, i.e. affine connection from \mathbf{p} , corresponds to data on $\mathfrak{g}/\mathfrak{q}$ invariant under $Q_0 = G_0 \cap Q$. Similarly, objects related to the choice of a reduced scale, i.e. special affine connection from \mathbf{p} , correspond to data invariant under $G_0^{ss} \cap Q$.

Proposition A.2. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal structure of signature (n, n) associated to an n -dimensional projective structure (M, \mathbf{p}) via the Fefferman-type construction. Then any metric in \mathbf{c} corresponding to a reduced scale is a Patterson–Walker metric.*

Proof. Within the proof we refer to explicit matrix realisations from Appendix B.1.

The conformal structure on the Fefferman space \widetilde{M} corresponds to the canonical \widetilde{P} -invariant conformal class of inner products on $\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{p}}$. The embedding $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R}) \hookrightarrow \mathfrak{so}(n+1, n+1) = \widetilde{\mathfrak{g}}$ in (71) induces an isomorphism $\mathfrak{g}/\mathfrak{q} \cong \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{p}}$ of Q -modules. Under this identification, the conformal class on $\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{p}}$ determines a Q -invariant conformal class on $\mathfrak{g}/\mathfrak{q}$ that we need to express explicitly. On that account, elements in $\mathfrak{g}/\mathfrak{q}$ will be represented by matrices

of the form

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ X & * & * \\ w & Y^t & -\frac{z}{2} \end{pmatrix}, \quad (63)$$

where $z, w \in \mathbb{R}$ and $X, Y \in \mathbb{R}^{n-1}$. Now it turns out that the conformal class on $\mathfrak{g}/\mathfrak{q}$ is represented by the quadratic form, whose value on the entry from (63) is

$$Y^t X - zw. \quad (64)$$

This can be either checked by a straightforward application of the above mentioned isomorphism, or alternatively, one can show that the only quadratic form on $\mathfrak{g}/\mathfrak{q}$, which changes conformally under the Q -action, is the one given by (64) up to a constant multiple. By the same reasoning it further follows that this quadratic form is invariant under the action of $G_0^{ss} \cap Q$. Altogether, any metric in the conformal class on \widetilde{M} determined be a reduced scale corresponds to the quadratic form (64) in suitable frame.

In order to show that any such metric is Patterson–Walker, we have to reinterpret the characterisation from definition A.1 in current terms. Firstly, the vertical subbundle $V \subset T\widetilde{M}$ corresponds to the Q -invariant subspace $f = \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ so that $V \cong \mathcal{G} \times_Q f$. According to the description in (63), f is given by $X = 0$ and $w = 0$. The canonical identification $V \cong T^*M(2)$ corresponds to an isomorphism $f \cong (\mathfrak{g}/\mathfrak{p})^*(2)$ of Q -modules. As before, we identify $(\mathfrak{g}/\mathfrak{p})^*(2)$ with $\mathfrak{p}_+(2)$ via the Killing form on \mathfrak{g} . It is an easy exercise to show that the desired isomorphism is provided by the mapping $f \rightarrow \mathfrak{p}_+(2)$ given by

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ 0 & * & * \\ 0 & Y^t & -\frac{z}{2} \end{pmatrix} \mapsto \begin{pmatrix} 0 & Y^t & -z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (65)$$

Secondly, it turns out there is a unique Q_0 -invariant subspace, h , which is complementary to f in $\mathfrak{g}/\mathfrak{q}$. According to the previous description, it is given by $Y = 0$ and $z = 0$. This subspace corresponds to the horizontal distributions induced by the linear connections from \mathfrak{p} so that $H \cong \mathcal{G}_0 \times_{Q_0} h$. The identification $H \cong TM$ corresponds to the obvious isomorphism $h \cong \mathfrak{g}/\mathfrak{p}$ of Q_0 -modules.

Now, both f and h are isotropic subspaces with respect to the inner product determined by (64), therefore the condition (a) from A.1 is satisfied. Furthermore, for any $v \in f$ and $u \in h$, the inner product of v and u coincides with the pairing of the corresponding element $v \in \mathfrak{p}_+(2)$ with $u \in \mathfrak{g}/\mathfrak{p}$ via the Killing form. Hence also the condition (b) from A.1 is satisfied. This completes the proof. \square

APPENDIX B. EXPLICIT MATRIX REALISATIONS AND TRACTOR FORMULAS

B.1. Explicit matrix realisations. Here we provide explicit realisations of relevant Lie algebras introduced in subsection 4.1 in terms of block matrices. We will consider the inner product h and the involution K in $\mathbb{R}^{n+1, n+1}$ given by the block matrices

$$h := \begin{pmatrix} 0 & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix} \quad \text{and} \quad K := \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{n+1} \end{pmatrix} \quad (66)$$

with respect to the standard basis (e_1, \dots, e_{2n+2}) . Then $E = \langle e_1, \dots, e_{n+1} \rangle$ and $F = \langle e_{n+2}, \dots, e_{2n+2} \rangle$ and the decomposition (34) can be visualised in the block terms as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \begin{pmatrix} E \otimes F & \Lambda^2 E \\ \Lambda^2 F & E \otimes F \end{pmatrix}. \quad (67)$$

For $\tilde{v} := e_1 + e_{2n+2}$, the Lie algebra $\tilde{\mathfrak{p}}$ of the parabolic subgroup $\tilde{P} \subset \tilde{G}$ is of the following block form

$$\tilde{\mathfrak{p}} = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -W^t & -b \\ X & B & V & W & C & -X \\ 0 & Y^t & c & b & X^t & 0 \\ \hline 0 & -Y^t & -d & -a & -X^t & 0 \\ Y & D & -Z & -U & -B^t & -Y \\ d & Z^t & 0 & -w & -V^t & -c \end{array} \right), \quad (68)$$

where the constituents in each block are again blocks of sizes 1, $n-1$, 1 along the diagonal, which are arbitrary except for

$$a - b = d - c, \quad C^t = -C, \quad D^t = -D.$$

The nilradical $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp$ is then of the form

$$\tilde{\mathfrak{p}}_+ = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -V^t & -a \\ 0 & 0 & V & V & 0 & 0 \\ 0 & 0 & a & a & 0 & 0 \\ \hline 0 & 0 & -a & -a & 0 & 0 \\ 0 & 0 & -U & -U & 0 & 0 \\ a & U^t & 0 & -w & -V^t & -a \end{array} \right). \quad (69)$$

A choice of Levi subalgebra $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{p}}$ determines a grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. We shall choose $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{p}}_{op}$, where $\tilde{\mathfrak{p}}_{op} \subset \tilde{\mathfrak{g}}$ is the stabiliser of the light-like vector e_{n+2} . Explicitly,

$$\tilde{\mathfrak{g}}_0 = \left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ X & B & V & 0 & C & -X \\ 0 & Y^t & c & 0 & X^t & 0 \\ \hline 0 & -Y^t & -a - c & -a & -X^t & 0 \\ Y & D & -Z & 0 & -B^t & -Y \\ a + c & Z^t & 0 & 0 & -V^t & -c \end{array} \right). \quad (70)$$

The embedding $i' : \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ of Lie algebras has got the following form

$$\mathfrak{sl}(n+1) \hookrightarrow \mathfrak{so}(n+1, n+1), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}. \quad (71)$$

The subgroup $Q = i^{-1}(\tilde{P})$ is contained in P , the stabiliser in G of $v = (\tilde{v})_E = e_1$; the inclusion of corresponding Lie algebras is

$$\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}} = \begin{pmatrix} a & U^t & w \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix} \subset \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix} = \mathfrak{p},$$

where $\text{tr}(A) = 0$ and $a + \text{tr}(B) + c = 0$. The standard projective grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$,

$$\mathfrak{g}_- = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix}, \quad \mathfrak{p}_+ = \begin{pmatrix} 0 & U^t & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (72)$$

is compatible with the previous conformal grading so that the reduced Lie subalgebra $\mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{g}_0$ coincides with the intersection of $\mathfrak{g}_0 \cap \tilde{\mathfrak{g}}_0$. Explicitly,

$$\mathfrak{q}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix} \quad (73)$$

where $\text{tr}(A) = 0$.

B.2. Further relations. Using the matrix realisations from B.1 and/or some simple direct arguments we can also show the following further relations between the various subspaces:

The image of the map $\varphi : (\mathfrak{g}/\mathfrak{p})^* \rightarrow (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ used in subsection 3.1 is identified with the space of linear forms on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$ annihilating $f = \mathfrak{p}/\mathfrak{q}$, the kernel of the projection $\mathfrak{g}/\mathfrak{q} \rightarrow \mathfrak{g}/\mathfrak{p}$. Since $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$, $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{p}}_+$, and $f \subset \mathfrak{g}/\mathfrak{q}$ is maximally isotropic, we may conclude that

$$\varphi(\mathfrak{p}_+) = \tilde{\mathfrak{p}}_+ \cap \ker s_F \cong f[-2], \quad (74)$$

where we also use the conventions from (13). Moreover, we note that

$$(\tilde{\mathfrak{p}}_+ \cap \ker s_F)_{E \otimes F} = \mathfrak{p}_+ \quad \text{and} \quad (\tilde{\mathfrak{p}} \cap \ker s_F)_{E \otimes F} = \mathfrak{p}, \quad (75)$$

which can easily be seen from the explicit matrices. In the same manner, one may conclude other useful facts like

$$\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2 \bar{F} \subset \tilde{\mathfrak{g}}_0, \quad [\tilde{\mathfrak{p}}_+, \Lambda^2 \bar{F}] = \varphi(\mathfrak{p}_+), \quad (76)$$

etc.

B.3. Explicit Formulas for the Adjoint Tractor Bundle. Before we start with the proof we collect some further formulas for the conformal adjoint tractor bundle: The standard representation of $\tilde{\mathfrak{g}}$ on $\mathbb{R}^{n+1, n+1}$ gives rise to the action $\bullet : \widetilde{\mathcal{AM}} \otimes \tilde{\mathcal{T}} \rightarrow \tilde{\mathcal{T}}$. Explicitly, this has the form

$$\begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ \beta_a \end{pmatrix} \bullet \begin{pmatrix} \nu \\ \omega_b \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho^r \omega_r - \varphi \nu \\ \mu_b^r \omega_r - \sigma \rho_b - \nu \beta_b \\ \beta^r \omega_r + \varphi \sigma \end{pmatrix}, \quad (77)$$

cf. (14) and (19). Considering the adjoint representation of $\tilde{\mathfrak{g}}$, we obtain the action $\bullet : \widetilde{\mathcal{AM}} \otimes \widetilde{\mathcal{AM}} \rightarrow \widetilde{\mathcal{AM}}$ given explicitly by

$$\begin{pmatrix} \rho_a \\ \mu_{a_0 a_1} \mid \varphi \\ \beta_a \end{pmatrix} \bullet \begin{pmatrix} \bar{\rho}_a \\ \bar{\mu}_{a_0 a_1} \mid \bar{\varphi} \\ \bar{\beta}_a \end{pmatrix} = \begin{pmatrix} \rho^r \bar{\mu}_{ra} - \bar{\rho}^r \mu_{ra} - \varphi \bar{\rho}_a + \bar{\varphi} \rho_a \\ 2(-\beta_{[a_0} \bar{\rho}_{a_1]} + \bar{\beta}_{[a_0} \rho_{a_1]} - \mu_{[a_0}^r \bar{\mu}_{a_1]r}) \mid -\beta^r \bar{\rho}_r + \bar{\beta}^r \rho_r \\ \beta^r \bar{\mu}_{ra} - \bar{\beta}^r \mu_{ra} + \varphi \bar{\beta}_a - \bar{\varphi} \beta_a \end{pmatrix} \quad (78)$$

cf. (19). Further we shall need the explicit form of $\tilde{\partial}^* : \tilde{\mathbb{E}}_{[c_0 c_1]} \otimes \mathcal{AM} \rightarrow \tilde{\mathbb{E}}_c \otimes \mathcal{AM}$,

$$\tilde{\partial}^* \left(\begin{array}{c|c} \rho_{c_0 c_1 a} & \varphi_{c_0 c_1} \\ \mu_{c_0 c_1 a_0 a_1} & \\ \beta_{c_0 c_1 a} & \end{array} \right) = \left(\begin{array}{c|c} -2\mu^r{}_{cra} + 2\varphi_{ca} & \\ -4\beta_{c[a_0 a_1]} & -2\beta^r{}_{cr} \\ 0 & \end{array} \right). \quad (79)$$

The tractor bundle $S^2 \tilde{\mathcal{T}}$ has the structure

$$S^2 \tilde{\mathcal{T}} = \left(\begin{array}{c} \tilde{\mathbb{E}}[-2] \\ \tilde{\mathbb{E}}_a \\ \tilde{\mathbb{E}}_{(ab)_0}[2] \mid \tilde{\mathbb{E}} \\ \tilde{\mathbb{E}}_a[2] \\ \tilde{\mathbb{E}}[2] \end{array} \right).$$

The projection $\bullet : \Lambda^2 \tilde{\mathcal{T}} \otimes \Lambda^2 \tilde{\mathcal{T}} \rightarrow \Lambda^2 \tilde{\mathcal{T}}$ is given by (78) and using the left hand side of this display (with \bullet replaced by \odot), we can write the explicit form of \odot as

$$\left(\begin{array}{c|c} \rho_a & \\ \mu_{a_0 a_1} & \varphi \\ \beta_a & \end{array} \right) \odot \left(\begin{array}{c|c} \bar{\rho}_a & \\ \bar{\mu}_{a_0 a_1} & \bar{\varphi} \\ \bar{\beta}_a & \end{array} \right) = \left(\begin{array}{c|c} -\rho^r \bar{\rho}_r & \\ \rho^r \bar{\mu}_{ra} + \bar{\rho}^r \mu_{ra} - \varphi \bar{\rho}_a - \bar{\varphi} \rho_a & \\ -\mu_{(a}{}^r \bar{\mu}_{b)r} - \beta_{(a} \bar{\rho}_{b)} - \bar{\beta}_{(a} \rho_{b)} \mid 2\varphi \bar{\varphi} - \beta^r \bar{\rho}_r - \bar{\beta}^r \rho_r & \\ \beta^r \bar{\mu}_{ra} + \bar{\beta}^r \mu_{ra} + \varphi \bar{\beta}_a + \bar{\varphi} \beta_a & \\ -\beta^r \bar{\beta}_r & \end{array} \right). \quad (80)$$

APPENDIX C. DERIVING UNDERLYING PROPERTIES FROM TRACTORIAL DATA

In this appendix, we derive a number of algebraic and differential properties of the various tensorial and spinorial objects living on a split-signature conformal structure (\tilde{M}, \mathbf{c}) induced by the tractor 2-form \mathbf{K} , and the tractor-spinors \mathbf{s}_E and \mathbf{s}_F described in the main text.

C.1. Algebraic properties. Let $\mathbf{K} \in \Gamma(\Lambda^2 \tilde{\mathcal{T}})$ be a section of the adjoint tractor bundle over \tilde{M} . Then, given a choice a metric $g \in \mathbf{c}$, we can write

$$\mathbf{K} = \left(\begin{array}{c} \rho_a \\ \mu_{ab} \mid \varphi \\ k_a \end{array} \right) \in \left(\begin{array}{c} \tilde{\mathbb{E}}_a \\ \tilde{\mathbb{E}}_{ab}[2] \oplus \tilde{\mathbb{E}} \\ \tilde{\mathbb{E}}_a[2] \end{array} \right).$$

We shall now assume that \mathbf{K} , as an endomorphism of $\tilde{\mathcal{T}}$, squares to the identity. This condition imposes the following algebraic restrictions on the slots

$$\begin{aligned} k^a k_a &= 0, & \rho^a \rho_a &= 0, \\ \mu^a{}_b k^b &= \varphi k^a, & \mu^a{}_b \rho^b &= -\varphi \rho^a, \\ k^a \rho_a &= \varphi^2 - 1, & \mu^c{}_a \mu_{cb} &= g_{ab} + 2k_{(a} \rho_{b)}. \end{aligned} \quad (81)$$

In particular, k^a and ρ^a are light-like vector fields.

Further, $\tilde{\mathcal{T}}$ splits into the two ± 1 -eigentractor bundles $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$, i.e. for $V \in \Gamma(\tilde{\mathcal{F}})$ and $U \in \Gamma(\tilde{\mathcal{E}})$,

$$\mathbf{K}V = -V, \quad \mathbf{K}U = U. \quad (82)$$

Let $\mathbf{s}_E \in \Gamma(\tilde{\mathcal{S}}_-^*)$ and $\mathbf{s}_F \in \Gamma(\tilde{\mathcal{S}}_-)$ be the pure tractor-spinors annihilating $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ respectively. As in the setup of section 4.1, we shall choose the normalisation $\langle \mathbf{s}_E, \mathbf{s}_F \rangle = -\frac{1}{2}$ so that (33) holds. With respect to a metric $g \in \mathbf{c}$, we write $\mathbf{s}_E = \begin{pmatrix} \bar{\eta} \\ \eta \end{pmatrix} \in \begin{pmatrix} \tilde{\mathcal{S}}_+^*[-\frac{1}{2}] \\ \tilde{\mathcal{S}}_-^*[\frac{1}{2}] \end{pmatrix}$ and $\mathbf{s}_F = \begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix} \in \begin{pmatrix} \tilde{\mathcal{S}}_-[-\frac{1}{2}] \\ \tilde{\mathcal{S}}_+[\frac{1}{2}] \end{pmatrix}$. Here, the purity of \mathbf{s}_F entails that the weighted spinors χ , $\bar{\chi}$ are pure with maximally intersecting totally isotropic n -plane distributions provided $\bar{\chi}$ is non-zero, and similarly for \mathbf{s}_E .

Before we proceed, it will be convenient to introduce abstract index notation on spinor fields, cf. [PR86]. Sections of the irreducible spinor bundles $\tilde{\mathcal{S}}_+$ and $\tilde{\mathcal{S}}_-$ will be adorned with primed and unprimed upper-case Roman indices, so that the above spinors will be denoted $\chi^{A'}$ and $\bar{\chi}^A$, and similarly for dual bundles, i.e. $\bar{\eta}_{A'}$ and η_A . In abstract indices, the generators of the Clifford algebra of $(T\tilde{M}, g)$ will be denoted $\gamma_a^{B'}{}_{A'}$ and $\gamma_a^{B'}{}_{A'}$. To streamline notation further, we shall write

$$\begin{aligned} \chi_a^A &:= \gamma_a^{A'}{}_{B'} \chi^{B'} : \Gamma(T\tilde{M}) \rightarrow \Gamma(\tilde{\mathcal{S}}_-), & \bar{\chi}_a^{A'} &:= \gamma_a^{A'}{}_{B'} \bar{\chi}^B : \Gamma(T\tilde{M}) \rightarrow \Gamma(\tilde{\mathcal{S}}_+), \\ \eta_{aA'} &:= \eta_B \gamma_a^{B'}{}_{A'} : \Gamma(T\tilde{M}) \rightarrow \Gamma(\tilde{\mathcal{S}}_+^*), & \bar{\eta}_{aA'} &:= \bar{\eta}_{B'} \gamma_a^{B'}{}_{A'} : \Gamma(T\tilde{M}) \rightarrow \Gamma(\tilde{\mathcal{S}}_-^*). \end{aligned}$$

In particular, the totally isotropic n -plane distributions $\tilde{e} := \ker \eta$ and $\tilde{f} := \ker \chi$ correspond to the kernels of the maps $\eta_{aA'}$ and χ_a^A respectively. We also note that the purity of χ and $\bar{\chi}$ and the condition on maximal intersection can be expressed algebraically as

$$\chi^{aA} \chi_a^B = 0, \quad \bar{\chi}^{aA'} \bar{\chi}_{a'}^{B'} = 0, \quad \chi^{aA} \bar{\chi}_a^{B'} = -2 \chi^{B'} \bar{\chi}^A, \quad (83)$$

respectively, and similarly for η and $\bar{\eta}$.

Now, any vector v^a tangent to \tilde{f} can be expressed as $v^a = \alpha_A \chi^{aA}$ for some spinor α_A [HM88, BT89]. Similarly, any section $V \in \Gamma(\tilde{\mathcal{F}})$ and $U \in \Gamma(\tilde{\mathcal{E}})$ can be expressed as $h(V, X) = \langle \mathbf{v}, X \cdot \mathbf{s}_F \rangle$ and $h(U, X) = \langle \mathbf{u}, X \cdot \mathbf{s}_E \rangle$ for some $\mathbf{v} \in \Gamma(\tilde{\mathcal{S}}_+)$ and $\mathbf{u} \in \Gamma(\tilde{\mathcal{S}}_-^*)$, and for all $X \in \Gamma(\tilde{\mathcal{T}})$.

Using the Clifford action (17) in terms of the splitting, we find that the eigentractor equations (82) are equivalent to

$$\begin{aligned} k^a \chi_a^B &= 0, & k^a \eta_{aB'} &= 0, \\ k^a \bar{\chi}_a^{B'} - \sqrt{2}(\varphi + 1) \chi^{B'} &= 0, & k^a \bar{\eta}_{aB} + \sqrt{2}(\varphi - 1) \eta_B &= 0, \\ (\mu^a{}_b + \delta_b^a) \chi^{bB} + \sqrt{2} \bar{\chi}^B k^a &= 0, & (\mu^a{}_b - \delta_b^a) \eta_{aB'} - \sqrt{2} \bar{\eta}_{A'} k^a &= 0, \\ (\mu^a{}_b + \delta_b^a) \bar{\chi}^{bB'} + \sqrt{2} \chi^{B'} \rho^a &= 0, & (\mu^a{}_b - \delta_b^a) \bar{\eta}_A - \sqrt{2} \eta_A \rho^a &= 0, \\ \rho^a \chi_a^B + \sqrt{2}(\varphi - 1) \bar{\chi}^B &= 0, & \rho^a \eta_{aC'} - \sqrt{2}(\varphi + 1) \bar{\eta}_{C'} &= 0, \\ \rho^a \bar{\chi}_a^{B'} &= 0, & \rho^a \bar{\eta}_{aB} &= 0. \end{aligned} \quad (84)$$

By inspection, we immediately obtain

Lemma C.1. *For a given metric $g \in \mathbf{c}$, the following statements are equivalent:*

- (a) $\bar{\chi} = 0$;
- (b) $\mu^a_b v^b = -v^a$ for all $v^a \in \Gamma(\tilde{f})$;
- (c) $\rho_a = 0$ and $\varphi = -1$.

If any of these statements holds, then $\bar{\eta}$ is a pure spinor dual to χ , and μ^b_a acts as the identity on the annihilator of $\bar{\eta}$.

In particular, μ^b_a is a skew-symmetric endomorphism squaring to the identity, i.e. $\mu^a_c \mu^c_b = \delta^a_b$. Further, our choice of normalisation $\langle \mathbf{s}_E, \mathbf{s}_F \rangle = -\frac{1}{2}$ implies that $\chi^{A'} \bar{\eta}_{A'} = -\frac{1}{2}$, so that (84) leads to

$$k^a = 2\sqrt{2} \eta_A \chi^{aA}, \quad \mu_{ab} = 2 \bar{\eta}_{A'} \gamma_{ab}^{A'} \chi^{B'},$$

where $\gamma_{ab}^{A'} := \gamma_{[a}^{A'} \gamma_{b]}^C \gamma_{B'}^C$.

C.2. Differential properties. We shall now examine the consequences of further differential assumptions on the tractor objects \mathbf{s}_F and \mathbf{K} above. We shall first assume that \mathbf{s}_F is parallel with respect to the normal tractor connection. We shall then *add* the assumption that the projecting part k^a of \mathbf{K} is a conformal Killing field. Finally, in addition to these two assumptions, we shall impose an additional curvature condition on the Weyl tensor.

C.2.1. Pure twistor-spinor. Assume that \mathbf{s}_F is parallel with respect to the normal Cartan connection, i.e.

$$\tilde{\nabla}^{nor} \mathbf{s}_F = 0, \quad (85)$$

so that χ is a twistor spinor. By definition, (85) tells us that the curvature of the normal connection takes values in $(\tilde{\mathcal{E}} \otimes \tilde{\mathcal{F}})_0 \oplus \wedge^2 \tilde{\mathcal{F}}$.

As noted in [HM88, Tag12] and proved in [Tag15] (see Remark 5.17), purity of \mathbf{s}_F is equivalent to χ and $\bar{\chi}$ being pure with their associated distributions intersecting maximally. By [Tag12], this implies that the distribution \tilde{f} is integrable. Finally, it is shown in [HSSTZ], that there exist conformal scales for which χ is parallel, and thus the conditions given in Lemma C.1 hold. See also [Lis15] for a similar result.

C.2.2. Conformal Killing field. Let us now assume that in addition to (85), \mathbf{K} is parallel with respect to the prolongation connection, i.e.

$$\tilde{\nabla}_b^{nor} \mathbf{K} = k^a \tilde{\Omega}_{ab}^{nor}, \quad (86)$$

so that k^a is a conformal Killing vector field. By differentiating the eigen-tractor equation (82) for elements of $\tilde{\mathcal{F}}$, we immediately deduce from (85) and (86) that $i_k \tilde{\Omega}^{nor}$ takes values in $\Lambda^2 \tilde{\mathcal{F}}$. We shall not be too concerned with the tractor-spinor \mathbf{s}_E . It suffices to say that (85) and (86) imply

$$\tilde{\nabla}_b^{nor} \mathbf{s}_E - \frac{1}{8} k^a \tilde{\Omega}_{ab}^{nor} \cdot \mathbf{s}_E = 0, \quad (87)$$

as can be checked by direct substitution of (33). Re-expressing (87) in terms of the slots of \mathbf{s}_E , we obtain

$$\tilde{D}_a \bar{\eta}_{A'} - \frac{1}{\sqrt{2}} \tilde{P}_{ab} \gamma^{bB}_{A'} \eta_B - \frac{1}{8} k^b \tilde{W}_{bacd} \gamma^{cdB'}_{A'} \bar{\eta}_{B'} = 0, \quad (88)$$

$$\tilde{D}_a \eta_A - \frac{1}{\sqrt{2}} \gamma_a^{B'} \eta_{B'} - \frac{1}{8} k^b \tilde{W}_{bacd} \gamma^{cdB}_{A'} \eta_B = 0. \quad (89)$$

Proposition C.2. *Let \mathbf{s}_F and \mathbf{K} be a tractor 2-form and a pure tractor spinor satisfying (85) and (86) respectively. Then, in a scale such that χ is parallel,*

$$\tilde{D}_a k_b - \mu_{ab} - g_{ab} = 0, \quad (90)$$

$$\tilde{D}_a \mu_{bc} + 2\tilde{P}_{a[b} k_{c]} - k^d \tilde{W}_{dabc} = 0. \quad (91)$$

Further, the following integrability conditions hold

$$\tilde{P}_{ab} v^b = 0, \quad \text{for all } v^a \in \Gamma(\tilde{f}), \quad (92)$$

$$\tilde{Y}_{abc} k^c = 0, \quad (93)$$

$$k^a \tilde{W}_{abcd} v^c = 0, \quad \text{for all } v^a \in \Gamma(\tilde{f}), \quad (94)$$

$$\tilde{W}_{abcd} \mu^{cd} = 0, \quad (95)$$

$$\tilde{W}_{abcd} v^c w^d = 0, \quad \text{for all } v^a, w^a \in \Gamma(\tilde{f}), \quad (96)$$

$$v^a \tilde{Y}_{abc} = 0, \quad \text{for all } v^a \in \Gamma(\tilde{f}). \quad (97)$$

Proof. We assume the conditions given Lemma C.1. Writing out (86) in terms of its slots yields (90) and (91) together with

$$\tilde{P}_{ab} k^b = 0, \quad (98)$$

$$\tilde{P}_{ac} (\mu^c_b - g^c_b) - \tilde{Y}_{abc} k^c = 0. \quad (99)$$

Similarly, (85) yields $\tilde{P}_{ab} \gamma^b \chi = 0$, and thus (92). From (99), we now immediately get (93). Contracting equation (91) with any $v^a \in \Gamma(\tilde{f})$ and making use of the fact that \tilde{D}_a preserves \tilde{f} (since χ is parallel with respect to \tilde{D}_a) lead to (94).

In slots, the integrability condition $\tilde{\Omega}_{ab}^{nor} \bullet \mathbf{s}_F = 0$ reads as

$$\tilde{W}_{abcd} \gamma^{cd} \chi = 0, \quad (100)$$

$$\tilde{W}_{abcd} \gamma^{cdA} \bar{\chi}^B - 2\sqrt{2} \chi^{cA} \tilde{Y}_{cab} = 0. \quad (101)$$

Condition (100) implies both (95) and (96) as shown in [Tag12]. Condition (101) reduces to (97) under our assumption $\bar{\chi} = 0$. \square

C.2.3. Additional curvature condition. At this stage, we impose the additional curvature condition

$$\tilde{W}_{abcd} v^a w^d = 0, \quad \text{for all } v^a, w^a \in \Gamma(\tilde{f}). \quad (102)$$

If we let M denote the (local) leaf space of the foliation tangent to the distribution \tilde{f} , this condition allows the conformal structure \mathbf{c} on \tilde{M} to descend to a projective structure \mathbf{p} on M . In particular, by Proposition 6.2, scales in which χ is parallel are none other than the reduced scales involved in the Fefferman construction of the main text, and which provide a tighter connection between the geometry of (\tilde{M}, \mathbf{c}) and that of (M, \mathbf{p}) .

As a consequence of the twistor spinor equation on χ , we obtain

Lemma C.3. *Let $\chi \in \Gamma(\tilde{S}_{-[\frac{1}{2}]})$ be a pure twistor spinor with associated integrable n -plane distribution \tilde{f} . Suppose that (102) holds. Then*

$$v^a w^b \tilde{Y}_{abc} = 0, \quad \text{for all } v^a, w^a \in \Gamma(\tilde{f}). \quad (103)$$

Further, in a scale such that χ is parallel, we have

$$v^c \tilde{Y}_{abc} = 0, \quad \text{for all } v^a \in \Gamma(\tilde{f}). \quad (104)$$

Proof. From the integrability condition (101) of χ , we have

$$v^a \tilde{W}_{abcd} \gamma^{cdA} \tilde{\chi}^B = 2\sqrt{2} \chi^{cA} \tilde{Y}_{cab} v^a, \quad \text{for all } v^a \in \Gamma(\tilde{f}).$$

We must show that the LHS vanishes under the assumption (102), which is itself equivalent to $v^a \tilde{W}_{abcd} \in \Gamma(\tilde{f} \otimes \Lambda^2 \tilde{f})$ for all $v^a \in \Gamma(\tilde{f})$. It is therefore enough to show that $\phi_{ab} \gamma^a \gamma^b \tilde{\chi} = 0$ for any 2-form $\phi_{ab} \in \Gamma(\Lambda^2 \tilde{f})$.

Writing $\phi_{ab} = \phi_{AB} \chi_a^A \chi_b^B$ for some $\phi_{AB} = \phi_{[AB]}$, and using (83), we can immediately compute $\phi_{ab} \gamma^{abC} \tilde{\chi}^D = 0$, thus establishing (103).

To show (104), recall that in a reduced scale, for which χ is parallel, the distribution \tilde{f} is preserved by \tilde{D}_a . Henceforth, v^a and w^b will denote two arbitrary sections of \tilde{f} . Conditions (92) and (102) can be expressed as $\tilde{R}_{abcd} v^a w^d = 0$. Now, contract v^a and w^e into the Bianchi identity $\tilde{D}_{[a} \tilde{R}_{bc]de} = 0$ yields

$$0 = v^a w^e \tilde{D}_a \tilde{R}_{bcde}.$$

where we have used the fact that \tilde{f} is parallel. In particular, this implies that $0 = v^a \tilde{D}_a \tilde{\text{Ric}}_{bc}$, where $\tilde{\text{Ric}}_{bc} := \tilde{R}_{ab}{}^a{}_c$ is the Ricci curvature. To see this, we note that we can always regard \tilde{g}^{ab} as a section of $\tilde{f} \otimes \tilde{f}^*$. Since $\tilde{J} = 0$, we have that \tilde{P}_{ab} is proportional to $\tilde{\text{Ric}}_{ab}$ by a constant factor, and thus

$$v^a \tilde{D}_a \tilde{P}_{bc} = 0,$$

The claim (104) now follows from the definition of the Cotton tensor $\tilde{Y}_{cab} = 2\tilde{D}_{[a} \tilde{P}_{b]c}$ and another application of (92). \square

C.3. $\text{SL}(n)$ -connections. Let us assume as before that we are in a scale such that χ is parallel. Then, as we saw in Lemma C.1, the spinor fields χ and $\bar{\eta}$ are dual to each other, and determine an $\text{SL}(n)$ -structure on (\tilde{M}, \mathbf{c}) , and we have a splitting of the tangent bundle $T\tilde{M} = \tilde{f} \oplus \tilde{f}^*$, where $\tilde{f} = \ker \chi_a^A$ and $\tilde{f}^* = \ker \bar{\eta}_{aA}$. This will be particularly useful when considering local expressions of Patterson-Walker metrics given in [HSSTZ] describing the present construction. It thus makes sense to decompose the Weyl tensor into irreducible $\text{SL}(n)$ -parts. In particular, the integrability condition (100) yields

$$\begin{aligned} \tilde{W}_{abcd} &= \tilde{W}_{abcd}^{(2,2)} + \tilde{W}_{abcd}^{(1,3)} + \tilde{W}_{abcd}^{(0,4)} \\ &\in \left((S^2 \tilde{f}) \otimes_0 (S^2 \tilde{f}^*) \right) \oplus \left((\Lambda^2 \tilde{f}) \otimes_0 (\tilde{f}^* \otimes \tilde{f})_0 \right) \oplus \left(\Lambda^2 \tilde{f} \otimes \Lambda^2 \tilde{f} \right). \end{aligned}$$

Condition (102) now tells us that $\tilde{W}_{abcd}^{(2,2)} = 0$. The Weyl tensor therefore consists of the two $\text{SL}(n)$ -invariant pieces $\tilde{W}_{abcd}^{(1,3)}$ and $\tilde{W}_{abcd}^{(0,4)}$, where

$$\tilde{W}_{abcd}^{(0,4)} = -k^e \tilde{D}_{[a} \tilde{W}_{b]ecd}^{(1,3)} + k_{[c} \tilde{Y}_{d]ab}.$$

This follows from the fact that the integrability condition $\mathcal{L}_k \widetilde{W}_{bcd}^a = 0$ for k^a satisfying (90) reads as

$$k^e \widetilde{D}_e \widetilde{W}_{abcd} = 2 \mu_{[a}^e \widetilde{W}_{b]ecd} + 2 \mu_{[c}^e \widetilde{W}_{d]eab} - 2 \widetilde{W}_{abcd}, \quad (105)$$

and the Bianchi identity yields

$$k^e \widetilde{D}_e \widetilde{W}_{abcd} + 2 k^e \widetilde{D}_{[a} \widetilde{W}_{b]ecd} = 2 k_{[c} \widetilde{Y}_{d]ab}. \quad (106)$$

Now, let us define a connection \widetilde{D}_a'' by

$$\widetilde{D}_a'' V^b = \widetilde{D}_a V^b - \frac{1}{2} k^d \widetilde{R}_{da}{}^b{}_c V^c, \quad \text{for any } V^a \in \Gamma(T\widetilde{M}).$$

Using (99) and (88), we can immediately check

$$\widetilde{D}_a'' g_{bc} = 0, \quad \widetilde{D}_a'' \mu_{bc} = 0, \quad \widetilde{D}_a'' \chi^{A'} = 0, \quad \widetilde{D}_a'' \bar{\eta}_{A'} = 0,$$

i.e. \widetilde{D}_a'' is a metric-preserving $\text{SL}(n)$ -connection with curvature and torsion given by

$$\begin{aligned} \widetilde{R}''_{abdc} &= \widetilde{W}_{abcd} - \mu_{[c}^e \widetilde{W}_{d]eab} + 2 \mu_{[a|[c} \widetilde{\mathbf{P}}_{d]|b]} - 2 g_{[a|[c} \widetilde{\mathbf{P}}_{d]|b]}, \\ \widetilde{T}''_{ab}{}^c &= \frac{1}{2} \widetilde{R}_{ab}{}^c{}_d k^d \end{aligned}$$

Further, the integrability condition [Afi54, Der11] for \widetilde{D}'' to descend to an affine connection on the leaf space M of \widetilde{f} is satisfied, i.e.

$$v^a \widetilde{R}''_{abdc} w^c = 0, \text{ for all } v^a, w^a \in \Gamma(\widetilde{f}).$$

It is torsionfree on M , since for any smooth function f constant along \widetilde{f} , we have $2 \widetilde{D}_{[a}'' \widetilde{D}_{b]}'' f = -\widetilde{T}''_{ab}{}^c \widetilde{D}_c'' f = 0$ since $v^a \widetilde{T}''_{abc} = 0$ for any $v^a \in \Gamma(\widetilde{f})$.

Using our $\text{SL}(n)$ -invariant decomposition, we also have

$$\widetilde{R}''_{abdc} = \widetilde{W}_{abcd}^{(1,3)} - \mu_{[c}^e \widetilde{W}_{d]eab}^{(1,3)} + 2 \mu_{[a|[c} \widetilde{\mathbf{P}}_{d]|b]} - 2 g_{[a|[c} \widetilde{\mathbf{P}}_{d]|b]}.$$

We can therefore identify $\widetilde{W}_{abcd}^{(1,3)}$ and $\widetilde{\mathbf{P}}_{ab}$ with the projective Weyl tensor and the Rho tensor of the affine connection in the projective class \mathbf{p} associated to \mathbf{c} respectively.

Remark C.4. We can also define a connection \widetilde{D}_a' by

$$\widetilde{D}_a' V^b = \widetilde{D}_a V^b - \frac{1}{2} k^d \widetilde{W}_{da}{}^b{}_c V^c, \quad \text{for any } V^a \in \Gamma(T\widetilde{M}). \quad (107)$$

This is a metric-compatible connection, but is not an $\text{SL}(n)$ -connection since $\widetilde{D}_a' \mu_{bc} = 2 k_{[b} \widetilde{\mathbf{P}}_{c]a}$. Its curvature and torsion are given by

$$\begin{aligned} \widetilde{R}'_{abcd} &= \widetilde{W}_{abcd} - \mu_{[c}^e \widetilde{W}_{d]eab} + k_{[c} \widetilde{Y}_{d]ab} + 2 \delta_{[a}^c \widetilde{\mathbf{P}}_{b]d} - 2 g_{[a|[c} \widetilde{\mathbf{P}}_{d]|b]}^d, \\ \widetilde{T}'_{ab}{}^c &= \frac{1}{2} \widetilde{W}_{ab}{}^c{}_d k^d \end{aligned}$$

With reference to (60), it is then not too difficult to check that $\widetilde{\nabla}'$ is a Weyl connection for the induced Cartan connection $\widetilde{\omega}^{ind}$ with curvature given by (61) in a reduced scale.

C.4. Differential properties of the distributions \tilde{e} , \tilde{f} and k . We wish to investigate which geometric properties of the distributions \tilde{e} , \tilde{f} and $\tilde{e} \cap \tilde{f}$ defined by the spinor fields η , χ and vector field k respectively, satisfy.

We already know that the distribution \tilde{f} is integrable, which implies [HM88, Tag12] that the leaves of the foliation are totally geodetic with respect to any Levi-Civita connection \tilde{D}_c of any metric in the conformal class \mathbf{c} , i.e. $v^c \tilde{D}_c z^b \in \Gamma(\tilde{f})$, for all $v^a, z^a \in \Gamma(\tilde{f})$, or equivalently,

$$g_{ab} w^a v^c \tilde{D}_c z^b = 0, \quad \text{for all } v^a, w^a, z^d \in \Gamma(\tilde{f}).$$

For the conformal Killing vector field k^a , the foliation by curves is geodetic and shear-free, i.e.

$$k^b \tilde{D}_b k^a = \lambda k^a, \quad \mathcal{L}_k g_{ab} = \nu g_{ab} + k_{(a} \beta_{b)},$$

for some functions λ and ν , and 1-form β_a . But it is twisting, i.e. $\alpha \wedge d\alpha \neq 0$ where $\alpha = g(k, \cdot)$.

We now turn to the geometric properties of the distribution \tilde{e} annihilated by η . Unless $n = 2$, this distribution \tilde{e} is not integrable. If it were, its leaves would be totally geodetic, but (89) yields

$$g_{ab} w^a v^c \tilde{D}_c z^b = k^d v^c w^a z^b \tilde{W}_{abcd}, \quad \text{for all } v^a, w^a, z^d \in \Gamma(\tilde{e}).$$

However, the Bianchi identity $\tilde{W}_{[abc]d} = 0$ tells us that \tilde{e} satisfies a weaker condition, specifically,

$$g_{ab} \left(w^a v^c \tilde{D}_c z^b + v^a z^c \tilde{D}_c w^b + z^a w^c \tilde{D}_c v^b \right) = 0, \quad \text{for all } v^a, w^a, z^d \in \Gamma(\tilde{e}).$$

This condition is non-trivial except when $n = 2$, and fits into the classification of the intrinsic torsion of a totally isotropic n -plane distribution given in [Tag12].

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